Database alignment: fundamental limits and efficient algorithms

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Joint work with
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Motivation

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- Data junction often not directly possible
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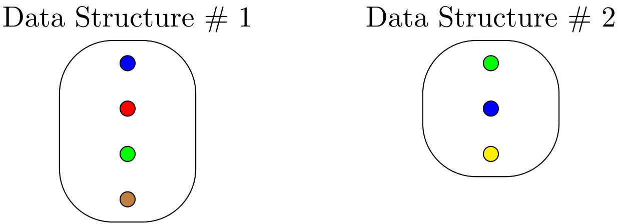
- Correlation among data
  - Possibility of non-obvious alignment and inference
- Risks on privacy
  - Crucial to understand conditions that allow or prevent privacy breaches

Alignment problem

The alignment problem studies the scenario where

- multiple data sources are present,



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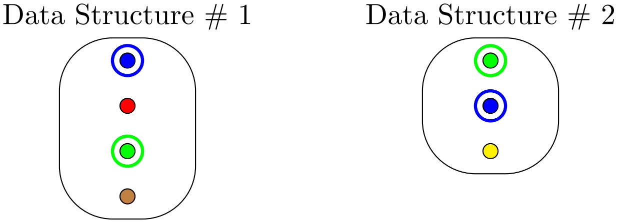

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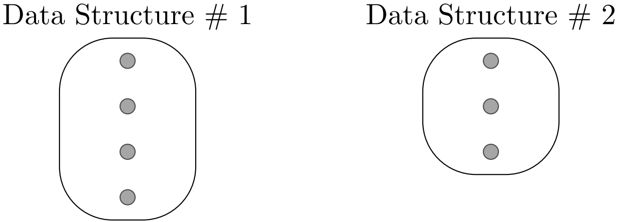

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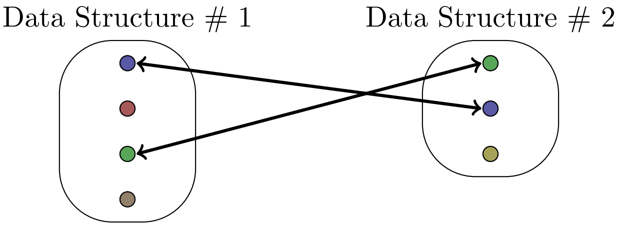
- it is possible to identify the correspondences *if* the correlation between data is strong enough.

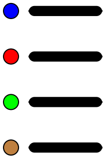


Alignment Problem

Structure of data

- Data associated to single users: database alignment
  E.g. medical records

Database



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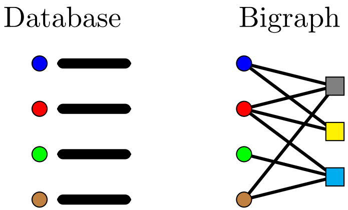

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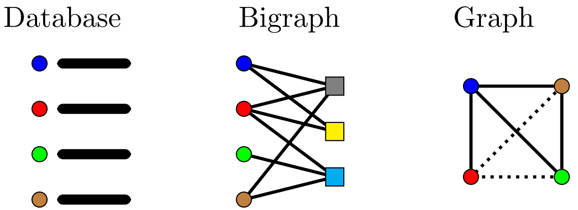
  E.g. connections on social media websites



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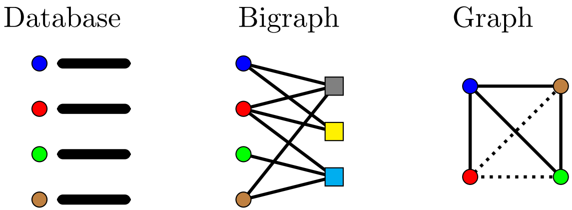
- Or any combination of these



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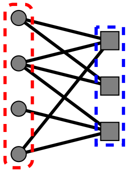

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Bipartite alignment: Movie ratings

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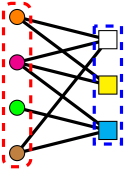
- IMDB user ratings
  Usernames , movie names, movie ratings



Illustrative example

- Common for IMDB users to register with full name

[1] Arvind Narayanan and Vitaly Shmatikov, "How To Break Anonymity of the Netflix Prize Dataset," S&P 2006

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- Netflix faced class action lawsuit and canceled sequal to competition.

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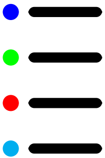
Database alignment

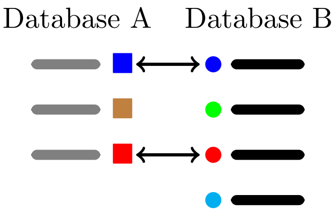
Correlated database model

Database: (unordered) set of features, each associated with a user



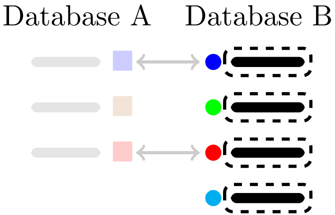
Correlated database model

Pair of databases with correlated data
Some users might not be present on both databases.



Correlated database model

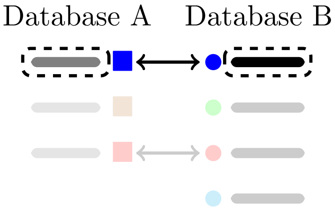
Features in a database are i.i.d.



independent
and
identically
distributed

Correlated database model

Features associated with the same user are jointly distributed.



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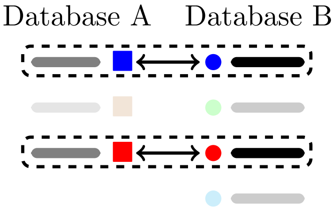

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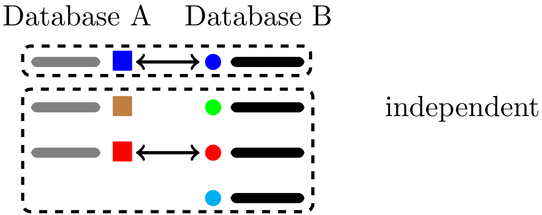
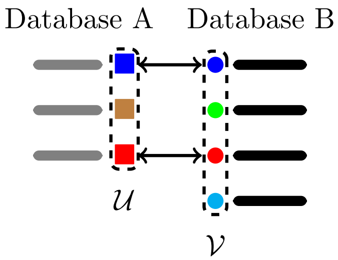
Correlated database model

Features of a user are independent from all other features.



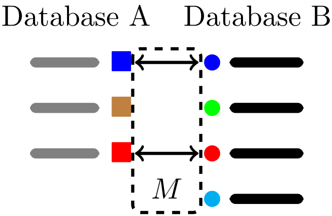
Likelihood function

$\mathcal{U}$ and $\mathcal{V}$: Sets of user identifiers.



Likelihood function

$\mathcal{U}$ and $\mathcal{V}$: Sets of user identifiers.
M: Mapping between identifiers

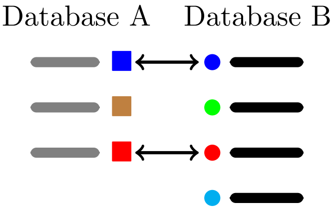


Likelihood function

$\mathcal{U}$ and $\mathcal{V}$: Sets of user identifiers.

Given some mapping M:

- $\mathcal{W}_M = \{({\color{blue}\blacksquare},{\color{blue}\bullet}), ({\color{red}\blacksquare},{\color{red}\bullet})\}$: pairs mapped by M
- $\mathcal{U}_M = \{{\color{blue}\blacksquare},{\color{red}\blacksquare}\}$: mapped users from A
- $\mathcal{V}_M = \{{\color{blue}\bullet},{\color{red}\bullet}\}$: mapped users from B



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- $X = A(\blacksquare)$: arbitrary feature from database A
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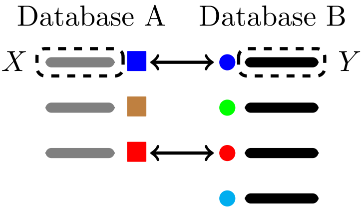

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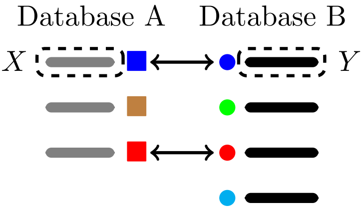
- f_X, f_Y: marginal pdfs (or pmfs) of features
- $f_{XY|M}$: joint pdf (or pmf) of features of a user, given M

The distributions are known.



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Then the log likelihood of databases A, B given M is

$$\sum_{(u,v)\in\mathcal{W}_M} \log f_{XY|M}\big(A(u), B(v)\big) + \sum_{u\in\mathcal{U}\setminus\mathcal{U}_M} \log f_X\big(A(u)\big) + \sum_{v\in\mathcal{V}\setminus\mathcal{V}_M} \log f_Y\big(B(v)\big)$$

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This can be rewritten as

$$\sum_{(u,v)\in\mathcal{W}_M} \log \frac{f_{XY|M}\big(A(u), B(v)\big)}{f_X\big(A(u)\big) f_Y\big(B(v)\big)} + \sum_{u\in\mathcal{U}} \log f_X\big(A(u)\big) + \sum_{v\in\mathcal{V}} \log f_Y\big(B(v)\big)$$

Maximum likelihood estimation

$$\sum_{(u,v)\in\mathcal{W}_M} \log \frac{f_{XY|M}\big(A(u),B(v)\big)}{f_X\big(A(u)\big)\,f_Y\big(B(v)\big)}$$
$$+ \sum_{u\in\mathcal{U}} \log f_X\big(A(u)\big)$$
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The last two terms do not depend on M.

We only need to consider the first term to maximize likelihood.

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The summation can be written as an inner product:

$$\sum_{(u,v)\in\mathcal{W}_{\color{blue}M}} \log \frac{f_{XY|M}\big({\color{red}A}(u),{\color{red}B}(v)\big)}{f_X\big({\color{red}A}(u)\big) f_Y\big({\color{red}B}(v)\big)} = \langle {\color{red}\mathbf{G}}, {\color{blue}\mathbf{M}} \rangle$$

Maximum likelihood estimation

- **G** is a function of the random databases A and B and is computable.

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Maximum likelihood estimation

- $\mathbf{G}$ is a function of the random databases A and B and is computable.
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- Therefore maximum likelihood estimation is possible in polynomial time.

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- Equivalent to picking the max entry in a row of $\mathbf{G}$.

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Database alignment

Results

Finite alphabet databases

- Features in A and B take values from finite alphabets $\mathcal{X}$ and $\mathcal{Y}$.

[3] Daniel Cullina, Prateek Mittal and Negar Kiyavash, Fundamental limits of database alignment, ISIT 2018

Finite alphabet databases

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- I_2° is the **Bhattacharyya distance** between the distribution of the databases under m and under m'.

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  Mutual information between correlated features I_{XY}
  I_{XY} is equal to the **Bhattacharyya distance** between the distribution of databases for 'adjacent' mappings.

[4] Osman Dai, Daniel Cullina, and Negar Kiyavash, Database alignment with Gaussian features, AISTATS 2019

Gaussian databases

- Features in A and B take values in vector spaces.
- Features of a user are multivariate Gaussian.
- Critical information theoretical measure:
  Mutual information between correlated features I_{XY}
  I_{XY} is equal to the **Bhattacharyya distance** between the distribution of databases for 'adjacent' mappings.
  Asymptotic case: $|M| = |\mathcal{U}| = |\mathcal{V}| = n \to \infty$
- **Achievability**: [4] MLE finds the correct mapping with probability $1 - o(1)$ as long as

$$I_{XY} \geq 2 \log n + \omega(1) \text{ [4]}.$$

- **Converse**: [4] Any algorithm fails to find the correct mapping with probability $1 - o(1)$ if

$$I_{XY} \leq 2 \log n (1 - \Omega(1)) \text{ [4]}.$$

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Gaussian databases

- These results consider estimation a success only if it gives the exact mapping.

Gaussian databases

- These results consider estimation a success only if it gives the exact mapping.
- What if the estimate is not exact but the ratio of errors is vanishingly small?

Gaussian databases

- Exact alignment
  - **Achievability**: If $I_{XY} \geq 2\log n + \omega(1)$, then MLE finds the correct mapping with probability $1 - o(1)$ [4].
  - **Converse**: If $I_{XY} \leq 2\log n(1 - \Omega(1))$, then any algorithm fails to find the correct mapping with probability $1 - o(1)$ [4].

- Almost-exact alignment
  - **Achievability**: If $I_{XY} \geq \log n + \omega(1)$, then MLE makes $o(n)$ errors in expectation [4].
  - **Converse**: If $I_{XY} \leq \log n(1 - \Omega(1))$, then any algorithm makes $\Omega(n)$ errors in expectation [4].

[4] Osman Dai, Daniel Cullina, and Negar Kiyavash, Database alignment with Gaussian features, AISTATS 2019

Gaussian alignment

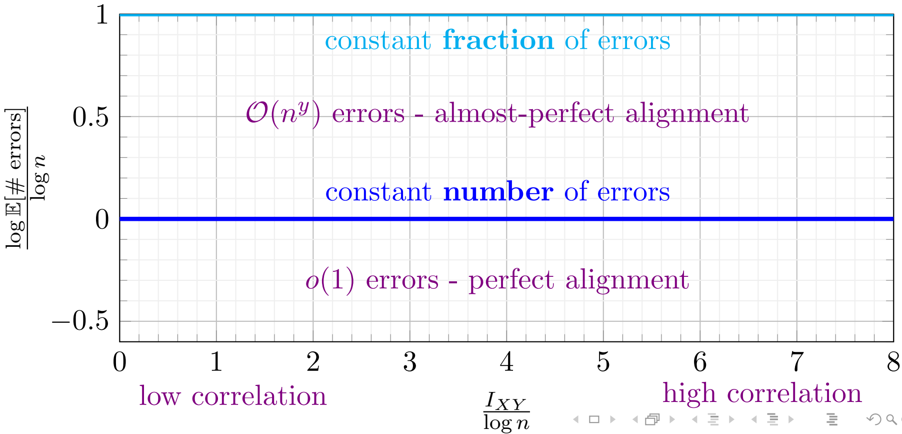
Asymptotic regime: $n \to \infty$

x-axis: mutual information

$I_{XY} = x \log n$

y-axis: error exponent

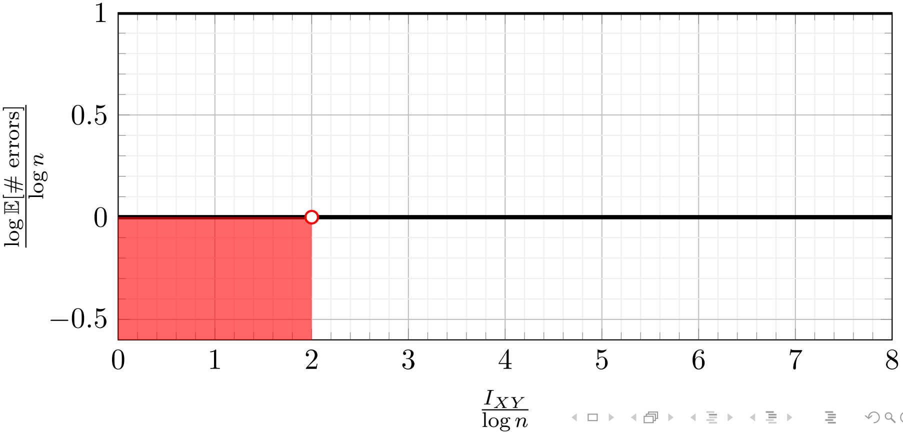
$\mathbb{E}[\#\text{errors}] = n^y$



Gaussian alignment

Converse:

- $I_{XY} \leq 2\log n(1-\Omega(1)) \implies \mathbb{E}[\#\text{errors}] \geq \Omega(1)$
- $x < 2 \implies y \geq 0$



Gaussian alignment

Converse:

- $I_{XY} \leq \log n(1 - \Omega(1)) \implies \mathbb{E}[\#\text{errors}] \geq \Omega(n)$
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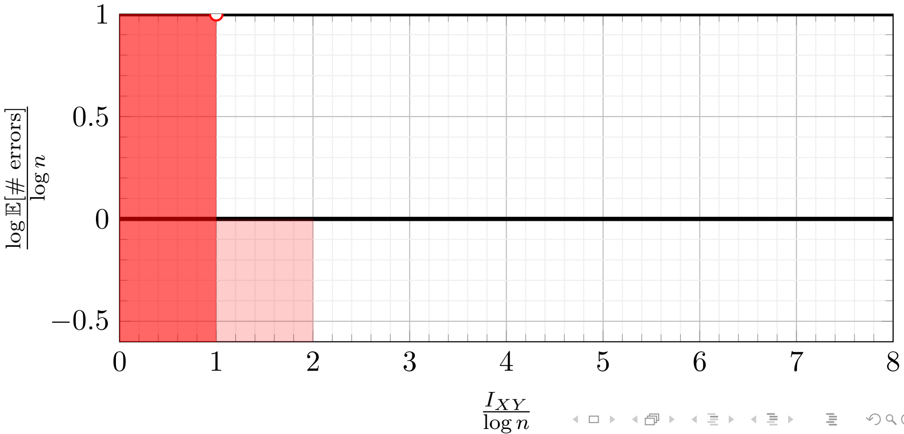
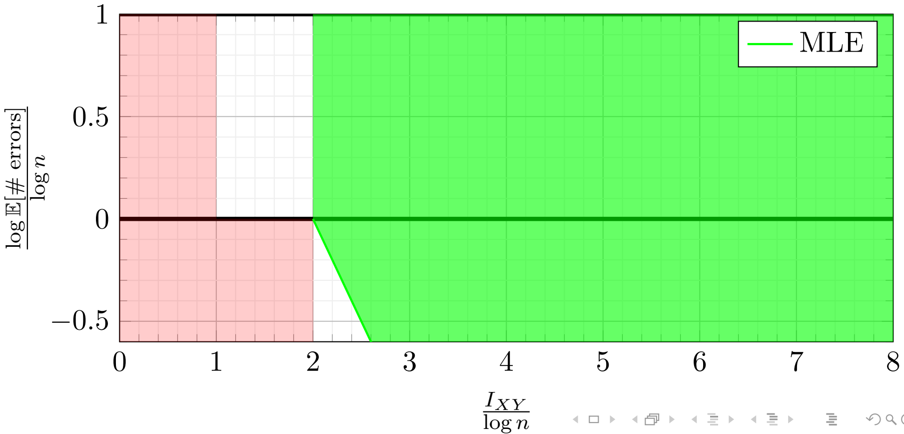
+



Gaussian alignment

Achievability (MLE - high correlation):

- $I_{XY} \geq 2\log n + \omega(1)$
  $\implies \mathbb{E}[\#\text{errors}] \leq 2\exp(2\log n - I_{XY})(1 + o(1))$
- $x > 2 \implies y \leq 2 - x$

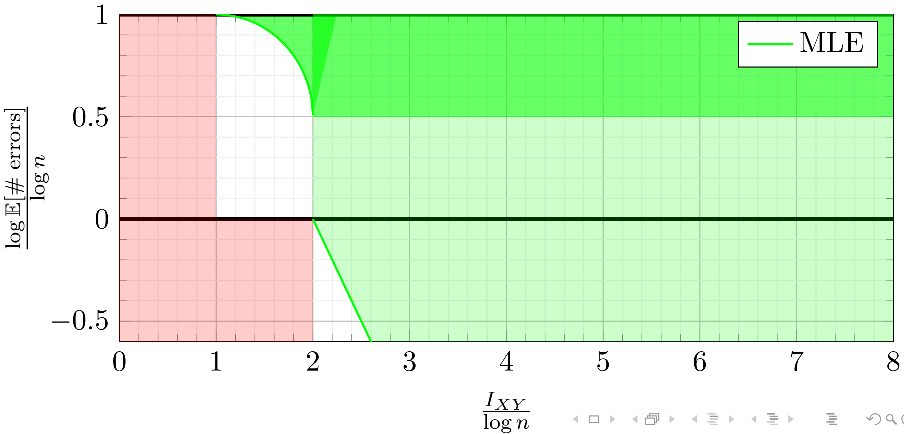


Gaussian alignment

Achievability (MLE - low correlation) [5]:

- $x > 1 \implies (x-1)^2 + (2y-1)^2 \leq 1$

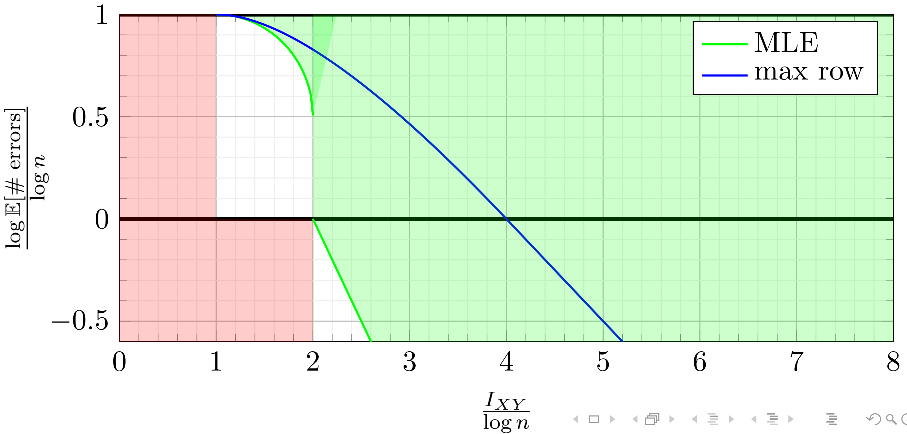
[5] Osman Dai, Daniel Cullina, and Negar Kiyavash, Achievability of nearly-exact alignment for correlated Gaussian databases, ISIT 2020



Gaussian alignment

Achievability (maximum row alignment):

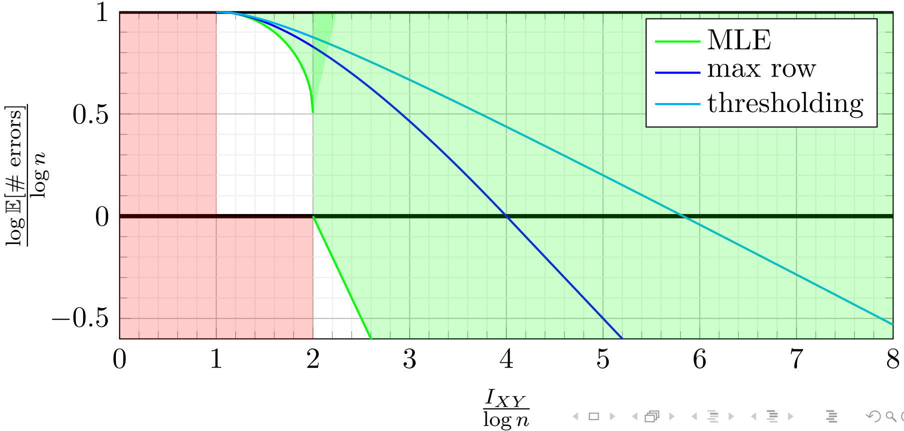
- High correlation: $x \geq 2 \implies y < 2 - \frac{x}{2}$
- Low correlation: $2 \geq x > 1 \implies y < 1 - (1 - \sqrt{x})^2$



Gaussian alignment

Achievability (thresholding):

- $x > 1 \implies y < 1 - \frac{x}{4} \cdot (1 - 1/x)^2$

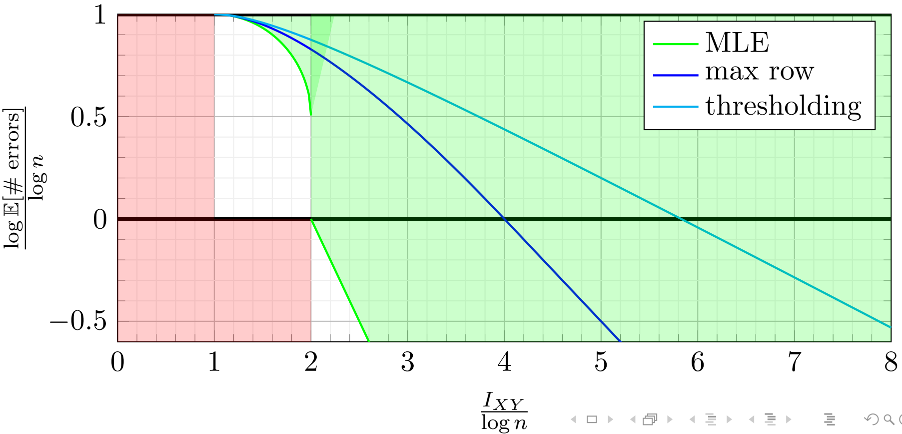


Gaussian alignment

All three algorithms make $o(n)$ errors right above the converse (Consistent with [6]).

---

[6] Farhad Shirani, Siddharth Garg, and Elza Erkip on discrete features, A Concentration of Measure Approach to Database De-anonymization, ISIT 2019



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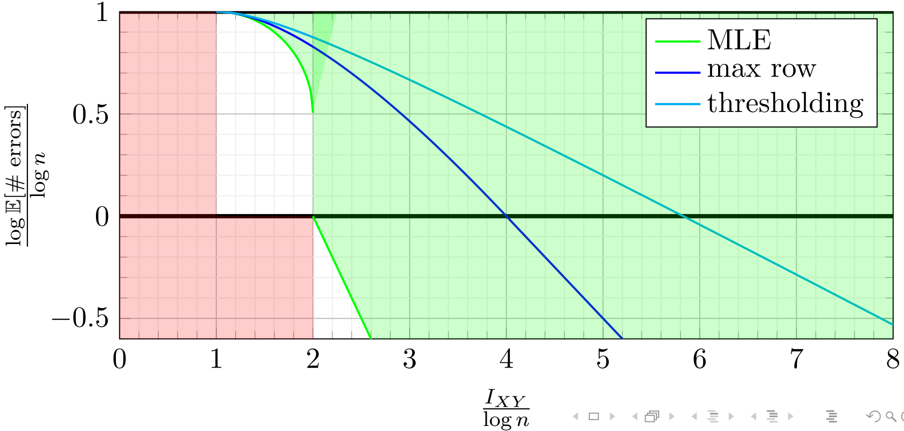


Database alignment

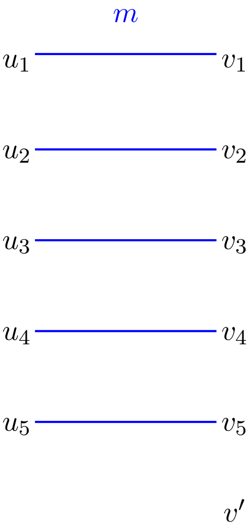
- Understanding the problem of database alignment provides insight on information theoretic quantities that characterize limits to alignment in general.

Outline of analysis

High-correlation achievability for MLE

Decomposition of misalignments

Graph representation

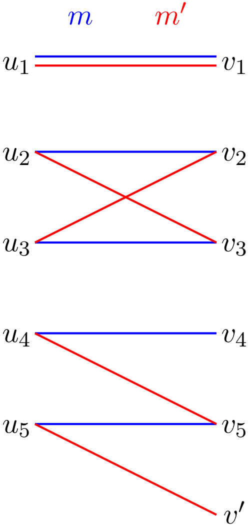


Matrix representation

$$\begin{array}{c}\mathbf{m}\\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}\end{array}$$

Decomposition of misalignments

Graph representation



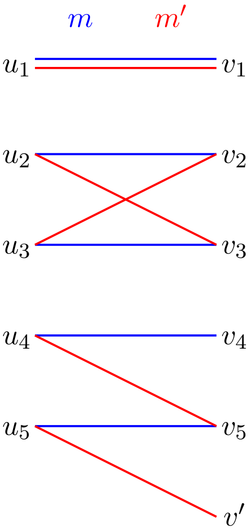
Matrix representation

$\mathbf{m}'$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Decomposition of misalignments

Graph representation



Matrix representation

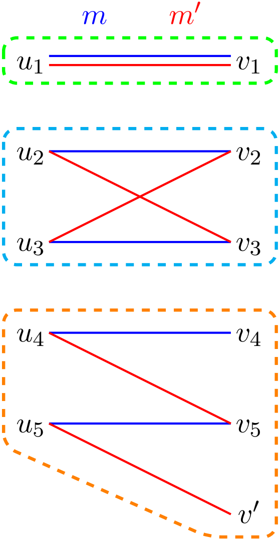
$$\mathbf{m'} - \mathbf{m}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 \end{bmatrix}$$

Decomposition of misalignments

Graph representation Matrix representation

Components $\longleftrightarrow$ Blocks



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Decomposition of misalignments

- Consider m' such that $\mathbf{m}' - \mathbf{m}$ induces multiple blocks:
  $\mathbf{m}' - \mathbf{m} = \begin{bmatrix} \mathbf{H}_1 & 0 \\ 0 & \mathbf{H}_2 \end{bmatrix}.$

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  $\mathbf{m'_1} - \mathbf{m} = \begin{bmatrix} \mathbf{H}_1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{m'_2} - \mathbf{m} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{H}_2 \end{bmatrix}$. These correspond to a 'partition' of the components that correspond to the blocks.

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- The difference in $\langle \mathbf{G}, \cdot \rangle$ decomposes:

$$\left\langle \mathbf{G}, \begin{bmatrix} \mathbf{H}_1 & 0 \\ 0 & \mathbf{H}_2 \end{bmatrix} \right\rangle = \left\langle \mathbf{G}, \begin{bmatrix} \mathbf{H}_1 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle + \left\langle \mathbf{G}, \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{H}_2 \end{bmatrix} \right\rangle$$

$$\langle \mathbf{G}, \mathbf{m}' - \mathbf{m} \rangle = \langle \mathbf{G}, \mathbf{m}'_1 - \mathbf{m} \rangle + \langle \mathbf{G}, \mathbf{m}'_2 - \mathbf{m} \rangle$$

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$$\langle \mathbf{G}, \mathbf{m'} - \mathbf{m} \rangle = \langle \mathbf{G}, \mathbf{m'}_1 - \mathbf{m} \rangle + \langle \mathbf{G}, \mathbf{m'}_2 - \mathbf{m} \rangle$$

- Therefore, if $\langle \mathbf{G}, \mathbf{m'} \rangle - \langle \mathbf{G}, \mathbf{m} \rangle = \langle \mathbf{G}, \mathbf{m'} - \mathbf{m} \rangle \geq 0$,

  then either $\langle \mathbf{G}, \mathbf{m'}_1 - \mathbf{m} \rangle \geq 0$
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  or $\langle \mathbf{G}, \mathbf{m'}_2 - \mathbf{m} \rangle \geq 0$.

- Then we can limit our attention to 'single-component'/'single-block' misalignments.

Decomposition of misalignments

- Let m be the true mapping and m' some false mapping.

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- Then all components are paths or cycles.

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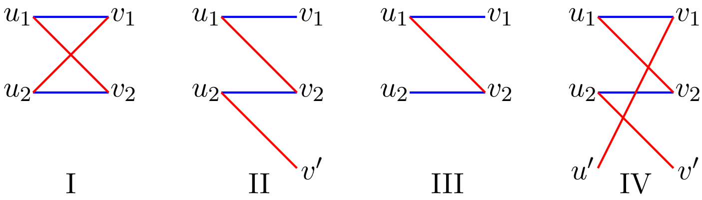

- I: cycle
  II: balanced path

  III: m-dominant path
  IV: m'-dominant path

Decomposition of misalignments

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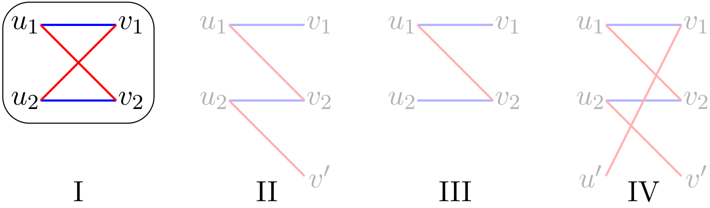

- I: cycle
  II: balanced path
  III: m-dominant path
  IV: m'-dominant path
- If $M = m$ matches all users, i.e. $|M| = |\mathcal{U}| = |\mathcal{V}|$, then there are no paths and all components are cycles.

Error bounds

Assume $|M| = |\mathcal{U}| = |\mathcal{V}|$, so only consider cycle-inducing false mappings.
If m' a false mapping that induces a **cycle of length 4**
with true mapping $M = m$, then the Chernoff bound gives us exactly:

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$$\mathbb{P}[\langle \mathbf{G}, \mathbf{m}' - \mathbf{m} \rangle \geq 0 | M = m] \leq \exp\left(-I_2^\circ\right)$$

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Assume $|M| = |\mathcal{U}| = |\mathcal{V}|$, so only consider cycle-inducing false mappings.
If m' a false mapping that induces a **cycle of length** 2δ with true mapping $M = m$,
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Henceforth we use to I without subscript to refer to either I_2° or I_{XY}.

Error bounds

- $|M| = |\mathcal{U}| = |\mathcal{V}| = n$
  There are $\binom{n}{\delta}(\delta-1)!$ false mappings m' that induce a cycle of length 2δ on m.

  $\binom{n}{\delta}(\delta-1)!$ is upper bounded by $\frac{n^\delta}{\delta}$.

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  The expected number of cycles of length 2δ contained in $\widehat{m} - m$ is bounded by

$$\frac{n^\delta}{\delta}\exp\left(-\delta\cdot\frac{I}{2}\right) = \frac{1}{\delta}\exp\left(-\frac{\delta}{2}(I - 2\log n)\right)$$

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- A cycle of length 2δ results in δ errors.
  The total number of errors in expectation is upper bounded by:

$$\sum_{\delta=2}^{n}\exp\left(-\frac{\delta}{2}(I - 2\log n)\right)$$

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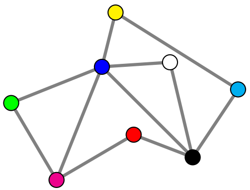
- If $I > 2\log n + \omega(1)$, this converges to $o(1)$.

Graph alignment

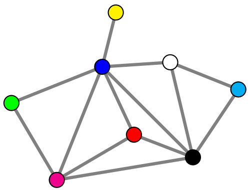
Graph alignment

In graph alignment, the data consists of edge information.

Network # 1



Network # 2



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The correlated graph model:

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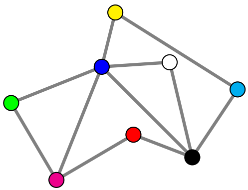
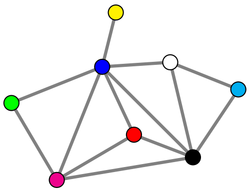


Network # 2



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The correlated graph model:

- Edges are i.i.d. random variables.

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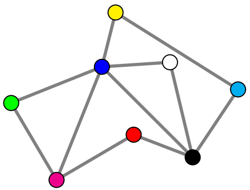
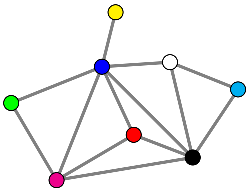


Network # 2



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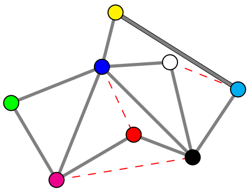
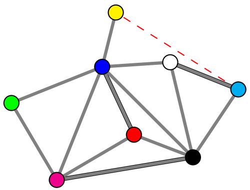


Network # 2



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Classical scenario: Edges are Bernouilli random variables taking values in {edge,non-edge}.

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Other settings of interest:

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  Allows modelling different types of connections in networks.

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- Continuous vector space.

  One case of particular interest is that of graphs with Gaussian weights [7], [8].

[7] Zhou Fan, Cheng Mao, Yihong Wu and Jiaming Xu, Spectral Graph Matching and Regularized Quadratic Relaxations I:The Gaussian Model, 2019

[8] Luca Ganassali, Sharp Threshold for Alignment of Graph Databases with Gaussian Weights 2020

Graph alignment - likelihood function

- f_X, f_Y: marginal pdfs (or pmfs) of edges in two graphs
- f_{XY}: joint pdf (or pmf) of correlated edge pair
- $\mathcal{W}_M \subset \mathcal{U} \times \mathcal{V}$: vertex pairs mapped by M

Let $\binom{\mathcal{S}}{2}$ denote set of all subsets of $\mathcal{S}$ of size 2.

Proxy for log-likelihood of graphs:

$$\sum_{\{(u_i,v_i),(u_j,v_j)\}\in\binom{\mathcal{W}_M}{2}} \log \frac{f_{XY}(A\{u_i,u_j\}, B\{v_i,v_j\})}{f_X(A\{u_i,u_j\}) f_Y(B\{v_i,v_j\})}$$

Graph alignment - likelihood function

- f_X, f_Y: marginal pdfs (or pmfs) of edges in two graphs
- f_{XY}: joint pdf (or pmf) of correlated edge pair
- $\mathcal{W}_M \subset \mathcal{U} \times \mathcal{V}$: vertex pairs mapped by M

Let $\binom{\mathcal{S}}{2}$ denote set of all subsets of $\mathcal{S}$ of size 2.

Proxy for log-likelihood of graphs:

$$\sum_{\{(u_i,v_i),(u_j,v_j)\} \in \binom{\mathcal{W}_M}{2}} \log \frac{f_{XY}(A\{u_i,u_j\}, B\{v_i,v_j\})}{f_X(A\{u_i,u_j\}) f_Y(B\{v_i,v_j\})} = \frac{1}{2} \overrightarrow{M}^\top \mathbf{G} \overrightarrow{M}$$

where $\mathbf{G} \in \mathbb{R}^{(\mathcal{U}\times\mathcal{V})\times(\mathcal{U}\times\mathcal{V})}$ information density matrix,
and $\overrightarrow{M} \in \{0,1\}^{(\mathcal{U}\times\mathcal{V})}$ encodes the mapping M.

Graph alignment - MLE

MLE for graph alignment is equivalent to the following optimization:

$$\max_{\vec{m}} \vec{m}^\top \mathbf{G} \vec{m} \qquad \text{s.t.} \qquad \begin{aligned} &\sum_v m_{(u,v)} = 1 && \forall u \in \mathcal{U} \\ &\sum_u m_{(u,v)} = 1 && \forall v \in \mathcal{V} \\ &\vec{m} \in \{0,1\}^{(\mathcal{U} \times \mathcal{V})} \end{aligned}$$

Databases vs Graphs

Database alignment

MLE given by linear optimization:

$\max \langle \mathbf{G}, \mathbf{m} \rangle = \operatorname{tr}(\mathbf{G}^\top \mathbf{m})$
over $\mathbf{m} \in \{0,1\}^{\mathcal{U}\times\mathcal{V}}$ with row and column sums equal to 1.

$\sim$ linear assignment problem
$\mathcal{O}(n^3)$

Graph alignment

MLE given by quadratic optimization:

$\max \overrightarrow{m}^\top \mathbf{G} \overrightarrow{m} = \operatorname{tr}(\mathbf{G} \overrightarrow{m} \overrightarrow{m}^\top)$
over $\overrightarrow{m} \in \{0,1\}^{(\mathcal{U}\times\mathcal{V})}$ with 'row' and 'column' sums equal to 1.

$\sim$ quadratic assignment problem
NP-hard

Graph alignment

Most well studied model: Correlated Erdős-Rényi

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  Difference of graphs $(V; E_A \setminus E_B)$ is Erdős-Rényi with avg. deg. $np(1-s)$

Generate graphs by independently generating edge random variable for each pair of vertices.

$$\Pr(\text{edge in } E_A \cap E_B) = ps$$
$$\Pr(\text{edge in } E_A \setminus E_B) = p(1-s)$$
$$\Pr(\text{edge in } E_B \setminus E_A) = p(1-s)$$
$$\Pr(\text{edge not in } E_A \cup E_B) = 1 - p(2-s)$$

Graph alignment

Results

Graph alignment

y-axis: strength of signal

0

average degree $\Theta(n^y)$ in intersection graph

$\frac{\log ps}{\log n}$

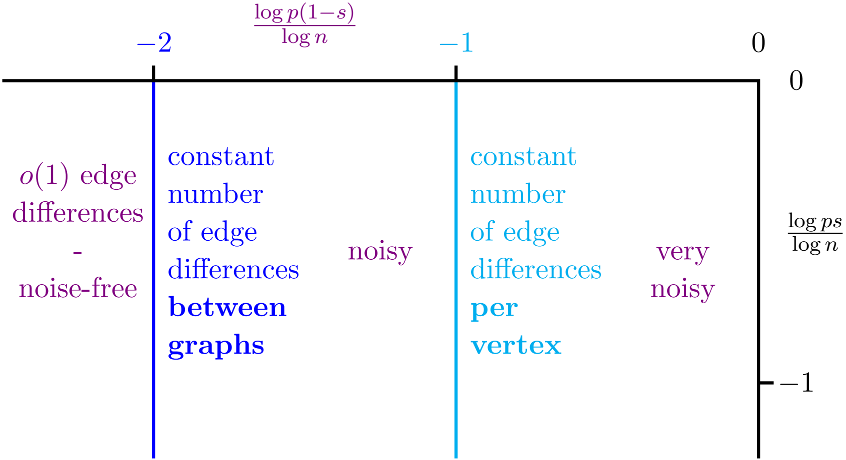
constant average degree in intersection graph

-1

average degree $o(1)$ in intersection graph

Graph alignment

x-axis: strength of noise $\quad$ y-axis: strength of signal



Graph alignment

Graphs are positively correlated if

$$\Pr(\text{in neither})\Pr(\text{in both}) > \Pr(\text{in } E_A \setminus E_B)\Pr(\text{in } E_A \setminus E_B)$$

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\Pr(\text{in neither})\Pr(\text{in both}) &> \Pr(\text{in } E_A \setminus E_B)\Pr(\text{in } E_A \setminus E_B) \\
\iff [1-p(2-s)]ps &> p^2(1-s)^2 \\
\iff s &> p \\
\iff \frac{y}{x} = \frac{\log ps/\log n}{\log p(1-s)/\log n} &> \frac{\log p^2}{\log p(1-p)}
\end{aligned}
$$

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$$

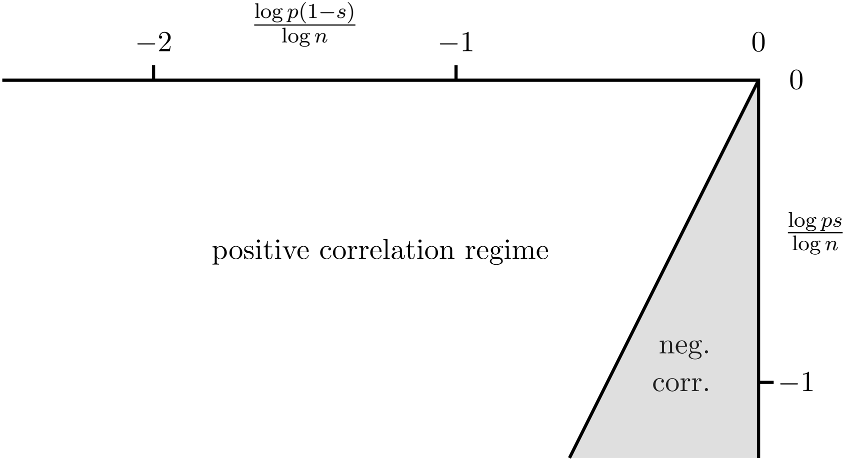
$$\iff \frac{y}{x} = \frac{\log ps/\log n}{\log p(1-s)/\log n} > \frac{\log p^2}{\log p(1-p)}$$

In the sparse regime $p \leq o(1)$, this corresponds to

$$\frac{y}{x} = \frac{\log ps/\log n}{\log p(1-s)/\log n} > 2.$$

Graph alignment

We only present results on positively correlated graphs.



Graph alignment

Noiseless case: $p(1-s) \leq o(n^2)$

Edward M. Wright - 1971 [9]

Sufficient and **necessary** condition for noiseless case:

Alignment possible with probability $1 - o(1)$ if and only if

$$np \geq \log n + \omega(1).$$

[9] Graphs on unlabeled nodes with a given number of edges, Acta Mathematica 1971

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The cut-off corresponds to the line

$$y = \frac{\log ps}{\log n} = -1 + \frac{\log \log n}{\log n} + \frac{\log s}{\log n} = -1 \pm o(1)$$

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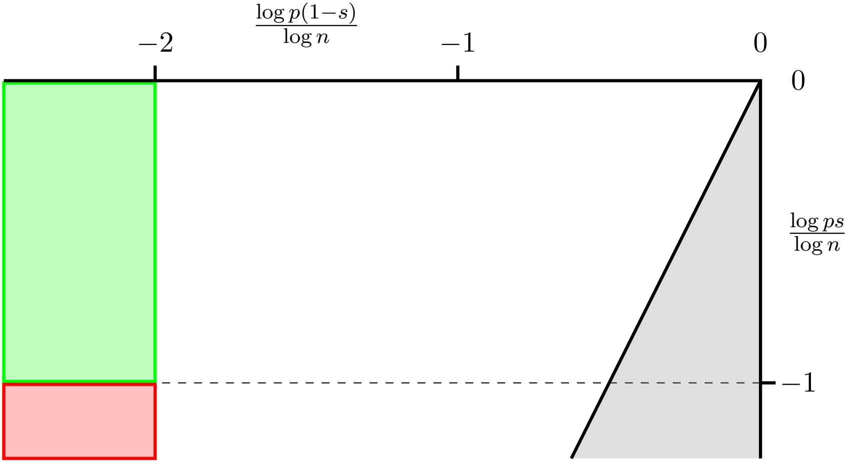
x-axis: strength of noise $\quad$ y-axis: strength of signal



Graph alignment

Noiseless case: $p(1-s) \leq o(n^2)$

Sufficient cond. for **polynomial-time** alignment:

Polynomial-time algorithms that achieve alignment with probability $1 - o(1)$ if $np \geq \log n + \omega(1)$

[10] Bla Bollobs, Distinguishing vertices of random graphs, North-Holland Mathematics Studies 1982

[11] Tomek Czajka and Gopal Pandurangan, Improved random graph isomorphism, Journal of Discrete Algorithms 2008

Graph alignment

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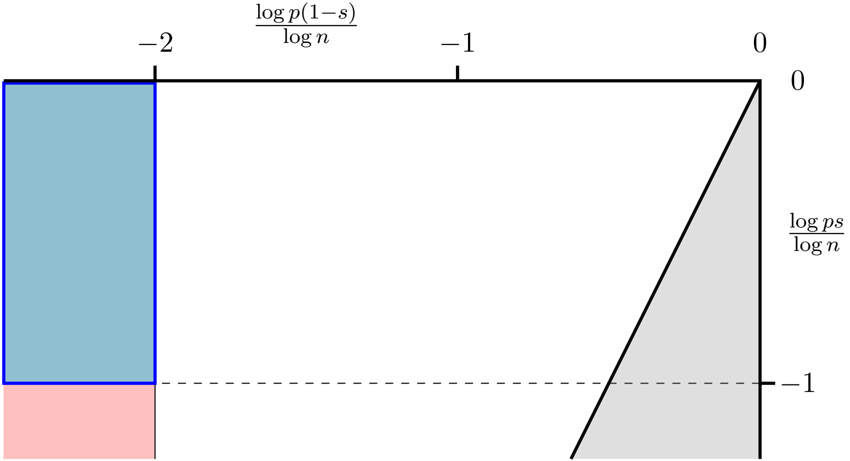

Graph alignment

Information theoretic bound formulations for sparse regime $p = o(1)$

- Pedram Pedarsani and Matthias Grossglauser 2011 [12]
  Sufficient condition: $nps\left(\frac{s}{2-s}\right) \geq 8\log n + \omega(1)$

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For small ps, the latter implies

$$2\log ps - \log p(1-s) > -\log n$$

or

$$2y - x = -1$$

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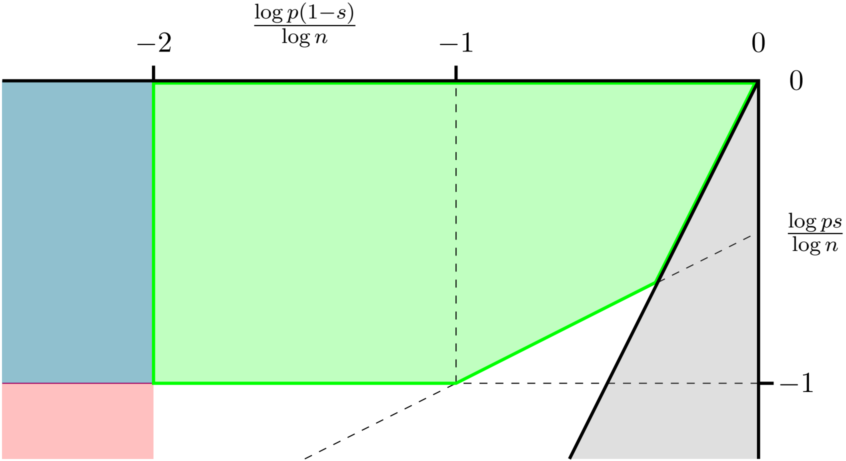

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Information-theoretic bounds for sparse regime $p = \mathcal{O}(1/\log n)$

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$$p(1-s) \leq \mathcal{O}(1/\log n) \text{ and } \frac{p(1-s)}{\sqrt{ps}} \leq \mathcal{O}(1/\log^{3/2} n)$$

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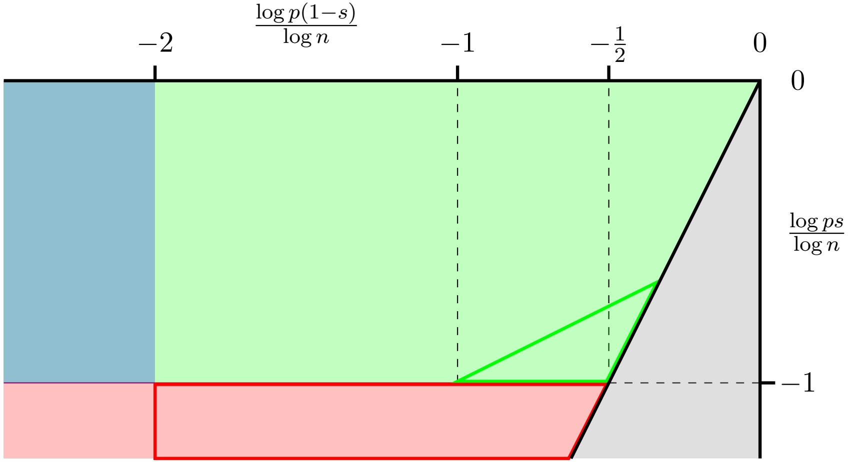
Graph alignment

x-axis: strength of noise $\qquad$ y-axis: strength of signal



Graph alignment

Regime of particular interest:

- Most algorithmic results focus on

$$-\log p \geq \Omega(\log n) \text{ and } -\log s \leq o(\log n)$$

  sparse graphs and s does not go to zero too quickly.

Graph alignment

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- Most algorithmic results focus on

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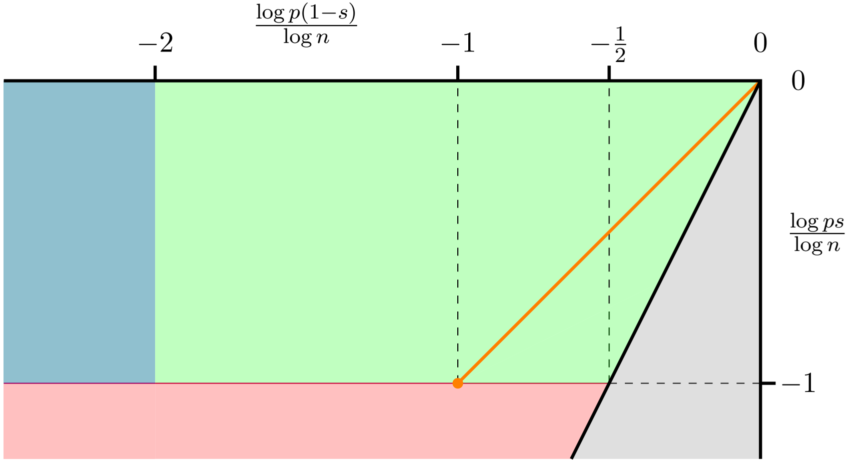
  sparse graphs and s does not go to zero too quickly.
- This entire regime of interest is contained on line $\frac{y}{x} = \frac{\log p + \log s}{\log p + \log(1-s)} = 1$.

Graph alignment

x-axis: strength of noise $\qquad$ y-axis: strength of signal



Graph alignment

Polynomial-time algorithms for exact alignment in the regime $(1-s) \le \mathcal{O}(1)$:

- Jian Ding, Zongming Ma, Yihong Wu and Jiaming Xu [15]
  $np \ge (\log n)^c$ and $(1-s) \le (\log n)^{-c}$
- Zhou Fan, Cheng Mao, Yihong Wu and Jiaming Xu [16]
  $np \ge (\log n)^c$ and $(1-s) \le (\log n)^{-c}$
- Cheng Mao, Mark Rudelson and Konstantin Tikhomirov [17]
  $np \ge (\log n)^c$ and $(1-s) \le (\log \log n)^{-c}$
- Cheng Mao, Mark Rudelson and Konstantin Tikhomirov [18]
  $n^{o(1)} \ge np \ge \log n(1+\varepsilon)$ and $(1-s) \le \min\{\text{constant}, \varepsilon\}$

---

[15] Efficient random graph matching via degree profiles, Probability Theory and Related Field 2021

[16] Spectral graph matching and regularized quadratic relaxations II: ErdosRenyi graphs and universality, 2019

[17] Random graph matching with improved noise robustness, Conference on Learning Theory 2021

[18] Exact Matching of Random Graphs with Constant Correlation, Conference on Learning Theory 2021

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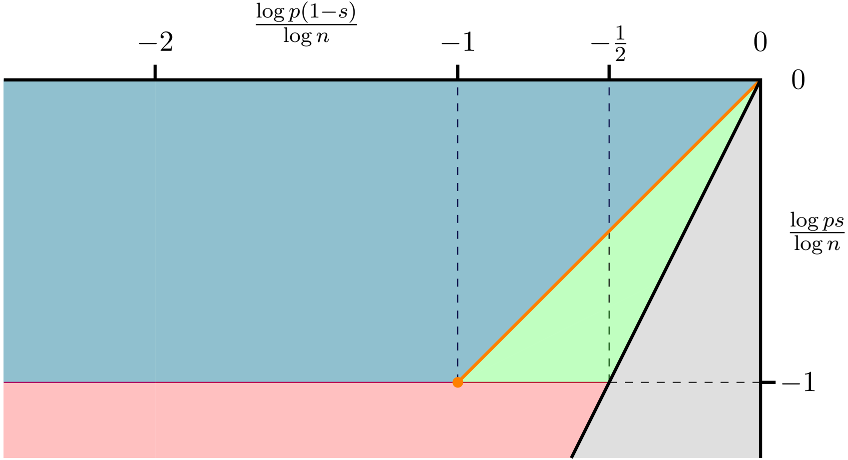

Graph alignment

All these polynomial-time algorithms have guarantees in the regime where s is bounded away from 0.

[19] (Nearly) Efficient Algorithms for the Graph Matching Problem on Correlated Random Graphs, Advances in Neural Information Processing Systems 2019

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Quasi-polynomial time algorithm for exact alignment:

- Boaz Barak, Chi-Ning Chou, Zhixian Lei, Tselil Schramm and Yueqi Sheng [19]
  $np \geq n^{o(1)}$ and $s \geq (\log n)^{-o(1)}$

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Unlike the polynomial algorithms, this algorithm allows $s \to 0$. This is still within the regime $-\log s \leq o(\log(n))$ and therefore does guarantee any region to the right of the $y = x$ line.

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Graph alignment - partial alignment

- **Exact alignment**:
  No misaligned vertices

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- **Almost-exact alignment**:
  Vanishing fraction of misaligned vertices

Graph alignment - partial alignment

- **Exact alignment**:
  No misaligned vertices
- **Almost-exact alignment**:
  Vanishing fraction of misaligned vertices
- **Partial alignment**:
  Constant fraction of misaligned vertices

Graph alignment - partial alignment

Necessary condition:

- Exact alignment [13]

$$nps > \log n(1 - \Omega(1))$$

- Almost-exact alignment [20]

$$nps > \mathcal{O}(1)$$

- Partial alignment [20]

$$nps > 1$$

[13] Daniel Cullina and Negar Kiyavash, Improved Achievability and Converse Bounds for Erdos-Renyi Graph Matching, Sigmetrics 2016

[20] Cullina, Daniel, Negar Kiyavash, Prateek Mittal and H. Vincent Poor, Partial Recovery of Erdos-Renyi Graph Alignment via k-Core Alignment, Sigmetrics 2020

[21] Luca Ganassali, Marc Lelarge and Laurent Massoulie, Impossibility of Partial Recovery in the Graph Alignment Problem, Annual Conference on Learning Theory 2021

Graph alignment - partial alignment

Sufficient condition:

- Exact alignment [13]

$$nps \geq \log n + \omega(1)$$

- Almost-exact alignment [20]

$$nps \geq \omega(1)$$

- Partial alignment [22]

$$nps \geq \max\left\{4, \frac{2\log n}{\log(s/p)}\right\}(1 + \mathcal{O}(1))$$

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[14] Daniel Cullina and Negar Kiyavash, Exact alignment recovery for correlated Erdos-Renyi graphs, 2017

[20] Cullina, Daniel, Negar Kiyavash, Prateek Mittal and H. Vincent Poor, Partial Recovery of Erdos-Renyi Graph Alignment via k-Core Alignment, Sigmetrics 2020

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Graph alignment - dense graphs

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Recent work improved the information theoretic bound for dense graphs with $p/s = \Theta(1)$.

Sufficient and necessary conditions:

- Daniel Cullina and Negar Kiyavash - 2016 [13] In the regime where $p \leq \mathcal{O}(1/\log n)$

$$nps\left(1-(1-s)\sqrt{p/s}\right)^2 \geq 2\log n + \omega(1)$$

$$nps > \log n(1-\Omega(1))$$

- Yihong Wu, Jiaming Xu and Sophie H. Yu - 2021 [21]

$$nps\left(1-\sqrt{p/s}\right)^2 \geq \log n(1+o(1))$$

$$nps\left(1-\sqrt{p/s}\right)^2 > \log n(1-o(1))$$

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[22] Yihong Wu, Jiaming Xu and Sophie H. Yu, Settling the Sharp Reconstruction Thresholds of Random Graph Matching, 2021

Thank you.