Flexibility can hurt dynamic matching system performance

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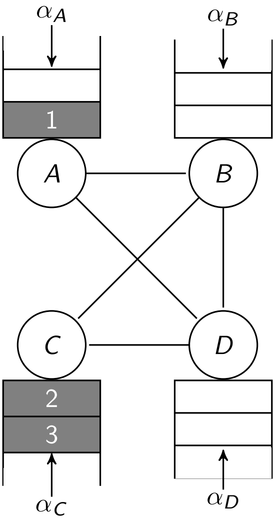
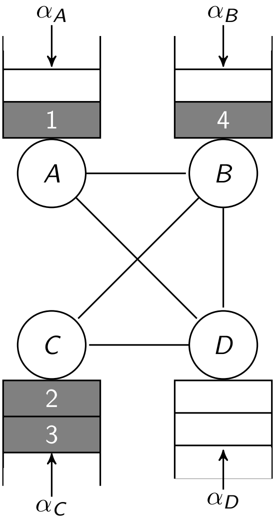
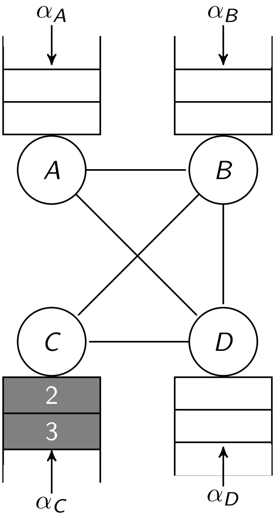
General stochastic matching model under FCFM policy



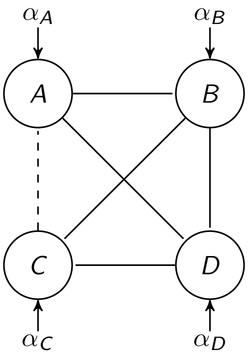
General stochastic matching model under FCFM policy



General stochastic matching model under FCFM policy



Contributions



- A closed-form expression for the expected total number of items remaining in the system.
- Sufficient conditions for the existence or the non-existence of a performance paradox in general stochastic matching models under heavy-traffic conditions.

Related work on general stochastic matching models

Stability

- [MM16] J. Mairesse and P. Moyal. *Stability of the stochastic matching model*. J. Appl. Probab., 2016.

FCFM

- [CKW09] R. Caldentey, E. Kaplan, G. Weiss. *FCFS infinite bipartite matching of servers and customers*. Adv. Appl. Probab., 2009.
- [AW12] I. Adan and G. Weiss. *Exact FCFS matching rates for two infinite multitype sequences*. Operations Research, 2012.
- [ABMW18] I. Adan, A. Bušić, J. Mairesse and G. Weiss. *Reversibility and further properties of FCFS infinite bipartite matching*. Math. Oper. Res., 2018.
- [MBM21] P. Moyal, A. Bušić and J. Mairesse. A product form for the general stochastic matching model. J. Appl. Probab., 2021.

Connected non-bipartite compatibility graph: $\mathcal{G} = (\mathcal{V}, \xi)$.
Arrivals of items of different classes follow independant Poisson processes, rate λ_i for class i.
After uniformization: discrete time model with at most one arrival per time step. Item class distribution: $\alpha = (\alpha_i)_{i \in \mathcal{V}}$ and α_0 for zero arrivals.
The policy used is First Come First Matched.
The dynamics of the system is modeled with a Markov chain $W = (W_t)_{t \in \mathbb{N}}$ where a state is represented by a word $w = w_1 \cdots w_q$.

A subset of nodes $\mathcal{I} \subseteq \mathcal{V}$ is called an *independent set* if there is no edge between any two nodes in $\mathcal{I}$, i.e. for any $i, j \in \mathcal{I}$, $(i, j) \notin \xi$.
Let $\mathbb{I}$ be the set of independent sets of $\mathcal{G}$.
Necessary and sufficient conditions for stability [MBM21]:

$$|\alpha_{\mathcal{I}}| < |\alpha_{\mathcal{E}(\mathcal{I})}|, \quad \forall \mathcal{I} \in \mathbb{I}.$$

where $|\alpha_V| = \sum_{i \in V} \alpha_i$, $\mathcal{E}(V) = \bigcup_{i \in V} \mathcal{E}(i)$ and
$\mathcal{E}(i) = \{j \in \mathcal{V} : (i, j) \in \xi\}$ for any $V \in \mathcal{V}$.

Expected value

Proposition

Let $\mathbb{E}[Q]$ be the expected stationary total number of items:

$$\mathbb{E}[Q] = \left(\sum_{\mathcal{I} \in \mathbb{I}} \sum_{\sigma \in \mathfrak{S}_{|\mathcal{I}|}} \sum_{l=1}^{|\mathcal{I}|} \frac{|\alpha_{\mathcal{E}(\mathcal{I}_l^\sigma)}|}{|\alpha_{\mathcal{E}(\mathcal{I}_l^\sigma)}| - |\alpha_{\mathcal{I}_l^\sigma}|} \prod_{k=1}^{|\mathcal{I}|} \frac{\alpha_{i_{\sigma(k)}}}{|\alpha_{\mathcal{E}(\mathcal{I}_k^\sigma)}| - |\alpha_{\mathcal{I}_k^\sigma}|} \right) \times \left(1 + \sum_{\mathcal{I} \in \mathbb{I}} \sum_{\sigma \in \mathfrak{S}_{|\mathcal{I}|}} \prod_{k=1}^{|\mathcal{I}|} \frac{\alpha_{i_{\sigma(k)}}}{|\alpha_{\mathcal{E}(\mathcal{I}_k^\sigma)}| - |\alpha_{\mathcal{I}_k^\sigma}|} \right)^{-1}$$

where $\mathfrak{S}_{|\mathcal{I}|}$ is the set of all permutations of $\mathcal{I}$ and $\mathcal{I}_k^\sigma = (i_{\sigma(1)}, \cdots, i_{\sigma(k)})$ the first k elements of the σ permutation of $\mathcal{I}$.

Heavy-traffic conditions

For any $\mathcal{I} \in \mathbb{I}$, denote by $|W_t|_{\mathcal{I}} = \sum_{i\in\mathcal{I}} |W_t|_i$, $t \geq 0$ and

$$\Delta_{\mathcal{I}} = |\alpha_{\mathcal{E}(\mathcal{I})}| - |\alpha_{\mathcal{I}}|.$$

Under FCFM policy, for any $t \geq 0$, we have

$$\mathbb{E}[|W_{t+1}|_{\mathcal{I}} - |W_t|_{\mathcal{I}}] \geq -\Delta_{\mathcal{I}},$$

Let

$$\bar{\delta} = \min_{\mathcal{I}\in\mathbb{I}} \Delta_{\mathcal{I}} = \min_{\mathcal{I}\in\mathbb{I}} (|\alpha_{\mathcal{E}(\mathcal{I})}| - |\alpha_{\mathcal{I}}|).$$

We select a bottleneck set $\hat{\mathcal{I}} \in \arg\min_{\mathcal{I}\in\mathbb{I}} \Delta_{\mathcal{I}}$ with the highest cardinality, i.e.

$$|\hat{\mathcal{I}}| = \max_{\mathcal{I}\in\mathbb{I} \text{ s.t. } \Delta_{\mathcal{I}}=\bar{\delta}} |\mathcal{I}|.$$

Heavy-traffic conditions

We define a parameterized family of item class distributions:

$$\alpha_i^\delta = \begin{cases} \alpha_i + \frac{\bar{\delta}}{2}\frac{\alpha_i}{|\alpha_{\hat{\mathcal{I}}}|} - \frac{\delta}{2}\frac{\alpha_i}{|\alpha_{\hat{\mathcal{I}}}|} & \text{if } i \in \hat{\mathcal{I}} \\ \alpha_i - \frac{\bar{\delta}}{2}\frac{\alpha_i}{|\alpha_{\mathcal{E}(\hat{\mathcal{I}})}|} + \frac{\delta}{2}\frac{\alpha_i}{|\alpha_{\mathcal{E}(\hat{\mathcal{I}})}|} & \text{if } i \in \mathcal{E}(\hat{\mathcal{I}}) \\ \alpha_i & \text{otherwise} \end{cases}$$

for all $0 < \delta \leq \bar{\delta}$, such that $\Delta_{\hat{\mathcal{I}}}^\delta = \delta$.

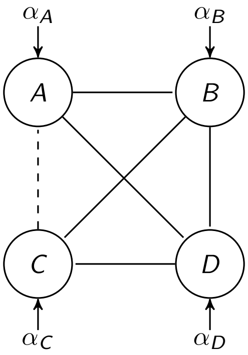
Performance paradox

Let $\widetilde{\mathcal{G}} = (\mathcal{V}, \widetilde{\xi})$ be the compatibility graph with the added edge (i^*, j^*), i.e $\widetilde{\xi} = \xi \cup \{(i^*, j^*)\}$.

Theorem

- *If $\hat{\mathcal{I}}$ has both i^* and j^* as neighbors, then there exists a performance paradox for δ sufficiently small.*
- *If $\hat{\mathcal{I}}$ contains i^* or j^* and $\mathcal{E}(\hat{\mathcal{I}}) \subsetneq \widetilde{\mathcal{E}}(\hat{\mathcal{I}})$, then there does not exist a performance paradox for δ sufficiently small.*

Example of a performance paradox



Let $\alpha_A = \alpha_C = 0.22$, $\alpha_B = 0.45$ and $\alpha_D = 0.11$.

We have $\bar{\delta} = \Delta_{\{B\}} = |\alpha_{\{A,C,D\}}| - |\alpha_B| = 0.1$. We define α^δ based on the bottleneck set $\hat{\mathcal{I}} = \{B\}$, for all $0 < \delta \leq 0.1$, i.e $\alpha_A^\delta = \alpha_C^\delta = 0.2 + \frac{\delta}{5}$, $\alpha_B^\delta = 0.5 - \frac{\delta}{2}$ and $\alpha_D^\delta = 0.1 + \frac{\delta}{10}$.

In addition, we have $\mathbb{E}[\widetilde{Q}] > \mathbb{E}[Q]$ for all $0 < \delta \leq 0.0818369$.