Simultaneously Achieving Sublinear Regret and Constraint Violations for Online Convex Optimization with Time-varying Constraints

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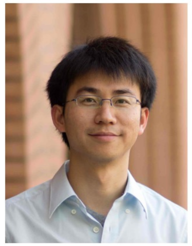
Performance 2021

Wenfei Wu



Longbo Huang

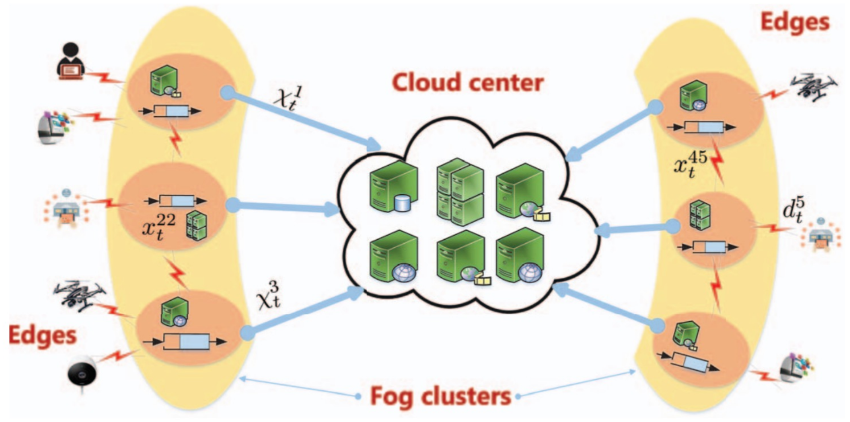
Zhixuan Fang

Outline

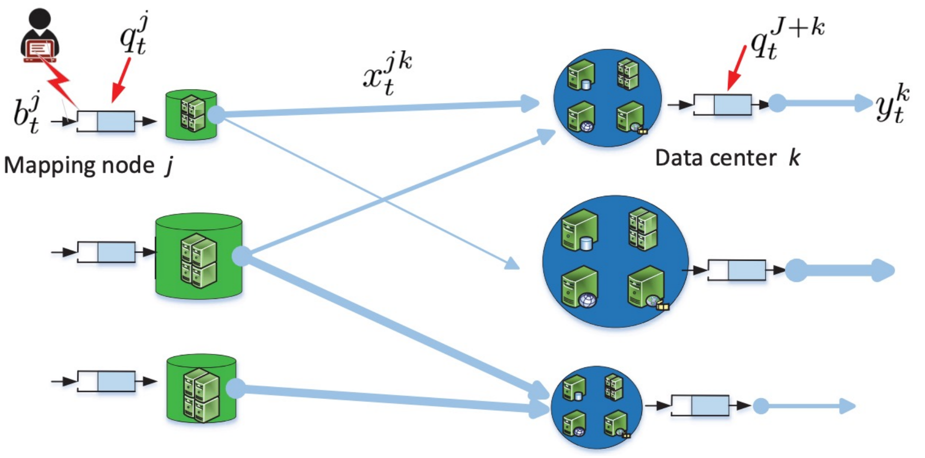
- Motivation
- Problem formulation
- Related works
- Algorithm
- Main results
- Simulation

Motivation

- Online network routing
- Online network resources allocation
- Online job scheduling
- Online fog computation offloading



A diagram of online fog computation offloading



A diagram of online network resources allocation

Problem formulation

- Model :
  - At round t, the agent makes a decision $x_t \in \chi$
  - Incurs a loss function f_t and a constraint function $\boldsymbol{g}_t$
  - f_t and $\boldsymbol{g}_t$ are time-varying, where $\boldsymbol{g}_t = [g_{t,1}(t), \dots, g_{t,k}(t)]^T$
  - Revealed after the decision making
- Goal :

$$\min_{\{x_t\}_{t=1}^{T}} \sum_{t=1}^{T} f_t(x_t), \ s.t. \ \sum_{t=1}^{T} \boldsymbol{g}_t(x_t) \le \boldsymbol{0}. \qquad \text{(P1)}$$

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$$\min_{\{x_t\}_{t=1}^{T}} \sum_{t=1}^{T} f_t(x_t), \ s.t. \ \sum_{t=1}^{T} \boldsymbol{g}_t(x_t) \leq \mathbf{0}. \qquad \text{(P1)}$$

- Challenging in the online setting

Related works

- [Chen et al., 2017, Chen et al., 2018, Cao and Liu, 2019, Chen and Giannakis, 2019]
  - Based on the modified online saddle-point (MOSP) methed
  - Assume the Slater condition holds
  - Cannot guarantee simultaneous sublinear regret (R) and constraint violations (C)

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- [Cao and Liu, 2019] :
  - The only one work that can achieve simultaneous R&C without the Slater condition
  - not parameter-free, i.e., the parameters in their algorithm require the prior information of the environments

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Non of the parameter-free methods can guarantee the sublinear regret and constraint violations simultaneously

Metrics

- Benchmark: per-slot minimizers $\{x_t^*\}$

$$x_t^* = \arg\min_{x \in \chi} \{f_t(x) | \boldsymbol{g}_t(x) \leq \boldsymbol{0}\}$$

- Dynamic regret

$$\mathrm{Regret}(T) = \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_t^*)$$

- Constraint violations

$$\mathrm{Vio}_k = \sum_{t=1}^{T} g_{k,t}(x_t)\,, k = 1,2, \ldots, K.$$

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- Constraint violations

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Note: dynamic regret is more meaningful in dynamic environments

Regularities

- Quantifying the temporal variations of functions sequence (or environment)
- Two main kinds of regularities [Chen et al., 2018, Yi et al., 2020]
  - Path-length: the accumulated variation of per-slot minimizers $\{x_t^*\}$

$$V_x = \sum_{t=2}^{T} ||x_t^* - x_{t-1}^*||$$

  - Function variation: the accumulated variation of consecutive constraints

$$V_{\boldsymbol{g}} = \sum_{t=2}^{T} \max_{x \in \chi} ||\boldsymbol{g}_t(x) - \boldsymbol{g}_{t-1}(x)||$$

Assumptions

- The feasible set χ is closed, convex, and bounded with diameter R, i.e.,
  - $||x - y|| \leq R, \forall x, y \in \chi$.
- f_t and $\boldsymbol{g}_t$ are convex, and bounded by F on χ, i.e.,
  - $\max_{x \in \chi}\{|f_t(x)|, ||\boldsymbol{g}_t(x)||\} \leq F, \forall t$.
- ∇f_t and $\nabla g_{k,t}$ are bounded by G over χ, i.e.,
  - $\max_{x \in \chi}\{||\nabla f_t(x)||, ||\nabla g_{k,t}(x)||\} \leq F, \forall k, t$.

Algorithm

Algorithm 1 VQB

Initialize: $\alpha_1, \gamma_0 > 0$, $\mathbf{g}_0 = \boldsymbol{\lambda}(0) = 0$, and $x_1 \in \chi$.
for round $t = 1 \ldots T-1$ **do**
  Update the dual iterate $\boldsymbol{\lambda}(t)$:
  $\boldsymbol{\lambda}(t) = \max\{\boldsymbol{\lambda}(t-1) + \gamma_{t-1}\mathbf{g}_{t-1}(x_t), -\gamma_{t-1}\mathbf{g}_{t-1}(x_t)\}$
  Update the primal iterate that satisfies:
  $x_{t+1} = \arg\min_{x \in \chi} \nabla f_t(x_t)^T (x - x_t) + [\boldsymbol{\lambda}(t) + \gamma_{t-1}\mathbf{g}_{t-1}(x_t)]^T (\gamma_t \mathbf{g}_t(x)) + \alpha_t \|x - x_t\|^2$
  Choose the action x_{t+1}
end for

- We introduces a sequence of dual variables $\{\lambda(t)\}$ (also called virtual queues)
  - To characterize the regret and constraint violations through the drift-plus-penalty expression
  - analyze the regret and constraint violations based on it

Algorithm

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  $\boxed{\text{Update the primal iterate that satisfies: } x_{t+1} = \arg\min_{x\in\chi} \nabla f_t(x_t)^T(x - x_t) + [\boldsymbol{\lambda}(t) + \gamma_{t-1}\boldsymbol{g}_{t-1}(x_t)]^T(\gamma_t \boldsymbol{g}_t(x)) + \alpha_t\|x - x_t\|^2}$
  Choose the action x_{t+1}
end for

- If there are no constraints $\boldsymbol{g}_t$ (i.e., $\boldsymbol{g}_t = \mathbf{0}$)
  - $\boldsymbol{\lambda}(t) = \mathbf{0}$

Algorithm

Algorithm 1 VQB

1: **Initialize:** $\alpha_1, \gamma_0 > 0$, $\boldsymbol{g}_0 = \boldsymbol{\lambda}(0) = 0$, and $x_1 \in \chi$.
2: **for** round $t = 1 \ldots T-1$ **do**
3: Update the dual iterate $\boldsymbol{\lambda}(t)$:
4: $\boldsymbol{\lambda}(t) = \max\{\boldsymbol{\lambda}(t-1) + \gamma_{t-1}\boldsymbol{g}_{t-1}(x_t), -\gamma_{t-1}\boldsymbol{g}_{t-1}(x_t)\}$
5: Update the primal iterate that satisfies:
6: $x_{t+1} = \arg\min_{x\in\chi} \nabla f_t(x_t)^T(x - x_t) + [\boldsymbol{\lambda}(t) + \gamma_{t-1}\boldsymbol{g}_{t-1}(x_t)]^T(\gamma_t \boldsymbol{g}_t(x)) + \alpha_t\|x - x_t\|^2$
7: Choose the action x_{t+1}
8: **end for**

- If there are no constraints $\boldsymbol{g}_t$ (i.e., $\boldsymbol{g}_t = \mathbf{0}$)
  - $\boldsymbol{\lambda}(t) = \mathbf{0}$
  - Update rule of x_{t+1} is equivalent to

$$x_{t+1} = \arg\min_{x\in\chi} \nabla f_t(x_t)^T(x - x_t) + \alpha_t\|x - x_t\|^2 = \Pi_\chi(x_t - \frac{1}{2\alpha_t}\nabla f_t(x_t)).$$

VQB reduces to OGD algorithm if there is no constraints

Algorithm

Update the primal iterate that satisfies:

$$x_{t+1} = \arg\min_{x \in \chi} \nabla f_t(x_t)^T (x - x_t) + [\boldsymbol{\lambda}(t) + \gamma_{t-1} \mathbf{g}_{t-1}(x_t)]^T (\gamma_t \mathbf{g}_t(x)) + \alpha_t \|x - x_t\|^2$$

- Define virtual queue backlogs:

$$\mathbf{Q}(\mathrm{t}) = \boldsymbol{\lambda}(t) + \gamma_{t-1} \boldsymbol{g}_{t-1}(x_t) = \max\{\boldsymbol{\lambda}(t-1) + 2\gamma_{t-1} \boldsymbol{g}_{t-1}(x_t), 0\}$$

- Define Lyapunov drift: $\Delta(t) = \frac{1}{2} ||\boldsymbol{Q}(t+1)||^2 - \frac{1}{2} ||\boldsymbol{Q}(t+1)||^2$

Algorithm

Update the primal iterate that satisfies:
$x_{t+1} = \arg\min_{x\in\chi} \nabla f_t(x_t)^T (x - x_t) + [\boldsymbol{\lambda}(t) + \gamma_{t-1}\mathbf{g}_{t-1}(x_t)]^T (\gamma_t \mathbf{g}_t(x)) + \alpha_t \|x - x_t\|^2$

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- Intuition: choose x_{t+1} to minimize an upper bound of the following expression

$$\underbrace{\Delta(t)}_{\text{drift}} + \underbrace{\nabla f_t(x_t)^T(x - x_t) + \alpha_t ||x - x_t||^2}_{\text{penalty}}.$$

Minimize penalty plus the Lyapunov drift

Algorithm

Update the primal iterate that satisfies:

$$x_{t+1} = \arg\min_{x\in\mathcal{X}} \nabla f_t(x_t)^T(x - x_t) + [\boldsymbol{\lambda}(t) + \gamma_{t-1}\boldsymbol{g}_{t-1}(x_t)]^T(\gamma_t \boldsymbol{g}_t(x)) + \alpha_t\|x - x_t\|^2$$

- Define virtual queue backlogs:

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- Define Lyapunov drift: $\Delta(t) = \frac{1}{2}||\boldsymbol{Q}(t+1)||^2 - \frac{1}{2}||\boldsymbol{Q}(t+1)||^2$
- Intuition: choose x_{t+1} to minimize an upper bound of the following expression

$$\underbrace{\Delta(t)}_{\text{drift}} + \underbrace{\nabla f_t(x_t)^T(x - x_t) + \alpha_t||x - x_t||^2}_{\text{penalty}}.$$

Since x_{t+1} has been determined, we replace $\boldsymbol{g}_t(x_{t+1})$ with $\boldsymbol{g}_t(x_{t+1})$ in $\Delta(t)$ and omit the constant term

Algorithm

Update the primal iterate that satisfies:

$$x_{t+1} = \arg\min_{x \in \chi} \nabla f_t(x_t)^T (x - x_t) + [\boldsymbol{\lambda}(t) + \gamma_{t-1}\mathbf{g}_{t-1}(x_t)]^T (\gamma_t \mathbf{g}_t(x)) + \alpha_t \|x - x_t\|^2$$

- Intuition: choose x_{t+1} to minimize an upper bound of the following expression (i.e., to minimize the penalty plus the Lyapunov drift)

$$\underbrace{\Delta(t)}_{\text{drift}} + \underbrace{\nabla f_t(x_t)^T (x - x_t) + \alpha_t ||x - x_t||^2}_{\text{penalty}}.$$

- The drift term $\Delta(t)$:
  - evaluate the constraint violations and is closed related to the virtual queues

Algorithm

Update the primal iterate that satisfies:

$$x_{t+1} = \arg\min_{x \in \chi} \nabla f_t(x_t)^T (x - x_t) + [\lambda(t) + \gamma_{t-1} \mathbf{g}_{t-1}(x_t)]^T (\gamma_t \mathbf{g}_t(x)) + \alpha_t \|x - x_t\|^2$$

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- The drift term $\Delta(t)$:
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- The regularization term $||x_t - x_{t-1}||^2$:
  - smoothen the difference between the coherent actions

Algorithm

Update the primal iterate that satisfies:
$x_{t+1} = \arg\min_{x \in \mathcal{X}} \nabla f_t(x_t)^T (x - x_t) + [\lambda(t) + \gamma_{t-1} \mathbf{g}_{t-1}(x_t)]^T (\gamma_t \mathbf{g}_t(x)) + \alpha_t ||x - x_t||^2$

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- The drift term $\Delta(t)$:
  - evaluate the constraint violations and is closed related to the virtual queues
- The regularization term $||x_t - x_{t-1}||^2$:
  - smoothen the difference between the coherent actions
- The remaining term $\nabla f_t(x_t)^T (x - x_t)$:
  - describes the optimization problem

Comparison

- Updating dual variables based on virtual queues
  - [Yu and Neely, 2020]: time-invariant constrains, static regret
  - [Qiu and Wei, 2020]: time-invariant constrains, static regret, parameter-dependent
- Our algorithm VQB differs in
  - Design a new way of involving instantaneous per-slot constraint violation into the virtual queues and decision sequence update for the time-varying constraints setting
  - The learning rates of our algorithm, i.e., α_t and γ_t are time-varying

Main results

- Without the slater condition:

Theorem 1:

(i) Set $\alpha_t = \sqrt{\frac{T}{R+\sum_{i\le t}||x_i^*-x_{i-1}^*||}}$, $\gamma_t = O(\frac{1}{\sqrt{2R}})$, we have

$$\mathrm{Regret}(T) \le O\left(\max\left\{\sqrt{TV_x}, V_g\right\}\right),$$

$$\mathrm{Vio}_k \le O\left(\max\left\{\sqrt{T}, V_g\right\}\right), \forall\, k = 1,2,\ldots,K.$$

(ii) Set $\alpha_t = \sqrt{\frac{T}{R+\sum_{i\le t}||x_i^*-x_{i-1}^*||}}$, $\gamma_t = O(\frac{1}{\sqrt{2R}}\frac{1}{\sqrt{t+1}})$, we have

$$\mathrm{Regret}(T) \le O\left(\sqrt{TV_x}\right),$$

$$\mathrm{Vio}_k \le O\left(\max\left\{T^{\frac{3}{4}}, V_g\right\}\right), \forall\, k = 1,2,\ldots,K.$$

Main results

- The variation of consecutive constraints is smooth across time (Slater condition) in many practical constrained OCO problems [Chen et al., 2017]
  - Question: Whether the Slater condition can lead to better bounds of constraint violations for VQB

Main results

- With the slater condition:

Theorem 2: Set $\alpha_t = O(\sqrt{T})$, $\gamma_t = O(T^{\frac{1}{4}})$, we have

$$\text{Regret}(T) \le O\left(\max\left\{\sqrt{TV_x}, \sqrt{TV_g}\right\}\right),$$

$$\text{Vio}_k \le O(1), \forall\, k = 1,2,\dots,K.$$

The $O(1)$ bound of constraint violations is achieved

Comparison

Reference	Regret(R)	Constraint violations(C)	Parameter-free	Slater condition-free[1]	Simultaneous sublinear R&C
[Chen et al., 2017]	$O(\max\{V_x T^a, V_g T^a, T^{1-a}\})$	$O(T^{1-a})$	✓	✗	✗
[Chen et al., 2018]	$O(T^{\frac{7}{8}} V_x)$	$O(\max\{T^{\frac{15}{16}}, T^{\frac{7}{8}} V_x\})$	✓	✓	✗
[Chen and Giannakis, 2019]	$O(V_x T^{\frac{3}{4}})$	$O(T^{\frac{3}{4}})$	✓	✗	✗
[Chen and Giannakis, 2019]	$O(V_x T^{\frac{1}{2}})$	$O(T^{\frac{1}{2}})$	✓	✗	✗
[Cao and Liu, 2018]	$O(V_x^{\frac{1}{2}} T^{\frac{1}{2}})$	$O(V_x^{\frac{1}{4}} T^{\frac{3}{4}})$	✗	✗	✓
Thm.1	$O(\max\{\sqrt{TV_x}, V_g\})$	$O(\max\{\sqrt{T}, V_g\})$	✓	✓	✓
Thm.1	$O(\sqrt{TV_x})$	$O(\max\{T^{\frac{3}{4}}, V_g\})$	✓	✓	✓
Thm.2	$O\max\{\sqrt{TV_x}, \sqrt{TV_g}\})$	$O(1)$	✓	✗	✓

- Sublinear regret and constraint violations simultaneously

Comparison

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[Chen et al., 2017]	$O(\max\{V_x T^a, V_g T^a, T^{1-a}\})$	$O(T^{1-a})$	✓	×	×
[Chen et al., 2018]	$O(T^{\frac{7}{8}} V_x)$	$O(\max\{T^{\frac{15}{16}}, T^{\frac{7}{8}} V_x\})$	✓	✓	×
[Chen and Giannakis, 2019]	$O(V_x T^{\frac{3}{4}})$	$O(T^{\frac{3}{4}})$	✓	×	×
[Chen and Giannakis, 2019]	$O(V_x T^{\frac{1}{2}})$	$O(T^{\frac{1}{2}})$	✓	×	×
[Cao and Liu, 2018]	$O(V_x^{\frac{1}{2}} T^{\frac{1}{2}})$	$O(V_x^{\frac{1}{4}} T^{\frac{3}{4}})$	×	×	✓
Thm.1	$O(\max\{\sqrt{TV_x}, V_g\})$	$O(\max\{\sqrt{T}, V_g\})$	✓	✓	✓
Thm.1	$O(\sqrt{TV_x})$	$O(\max\{T^{\frac{3}{4}}, V_g\})$	✓	✓	✓
Thm.2	$O\max\{\sqrt{TV_x}, \sqrt{TV_g}\})$	$O(1)$	✓	×	✓

- Matches the state-of-the-art dynamic regret bound $O(\sqrt{TV_x})$ in classic OCO, when the path-length of the benchmark sequence is V_x

Comparison

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[Chen et al., 2018]	$O(T^{\frac{7}{8}} V_x)$	$O(\max\{T^{\frac{15}{16}}, T^{\frac{7}{8}} V_x\})$	✓	✓	✗
[Chen and Giannakis, 2019]	$O(V_x T^{\frac{3}{4}})$	$O(T^{\frac{3}{4}})$	✓	✗	✗
[Chen and Giannakis, 2019]	$O(V_x T^{\frac{1}{2}})$	$O(T^{\frac{1}{2}})$	✓	✗	✗
[Cao and Liu, 2018]	$O(V_x^{\frac{1}{2}} T^{\frac{1}{2}})$	$O(V_x^{\frac{1}{4}} T^{\frac{3}{4}})$	✗	✗	✓
Thm.1	$O(\max\{\sqrt{TV_x}, V_g\})$	$O(\max\{\sqrt{T}, V_g\})$	✓	✓	✓
Thm.1	$O(\sqrt{TV_x})$	$O(\max\{T^{\frac{3}{4}}, V_g\})$	✓	✓	✓
Thm.2	$O \max\{\sqrt{TV_x}, \sqrt{TV_{\boldsymbol{g}}}\})$	$O(1)$	✓	✗	✓

- **Theorem 1 (case 1):** the regret and constraint violations bounds are all no worse than the state-of-the-art results when V_x is not too large

Comparison

Reference	Regret(R)	Constraint violations(C)	Parameter-free	Slater condition-free[1]	Simultaneous sublinear R&C
[Chen et al., 2017]	$O(\max\{V_x T^a, V_g T^a, T^{1-a}\})$	$O(T^{1-a})$	✓	✗	✗
[Chen et al., 2018]	$O(T^{\frac{7}{8}} V_x)$	$O(\max\{T^{\frac{15}{16}}, T^{\frac{7}{8}} V_x\})$	✓	✓	✗
[Chen and Giannakis, 2019]	$O(V_x T^{\frac{3}{4}})$	$O(T^{\frac{3}{4}})$	✓	✗	✗
[Chen and Giannakis, 2019]	$O(V_x T^{\frac{1}{2}})$	$O(T^{\frac{1}{2}})$	✓	✗	✗
[Cao and Liu, 2018]	$O(V_x^{\frac{1}{2}} T^{\frac{1}{2}})$	$O(V_x^{\frac{1}{4}} T^{\frac{3}{4}})$	✗	✗	✓
Thm.1	$O(\max\{\sqrt{TV_x}, V_g\})$	$O(\max\{\sqrt{T}, V_g\})$	✓	✓	✓
Thm.1	$O(\sqrt{TV_x})$	$O(\max\{T^{\frac{3}{4}}, V_g\})$	✓	✓	✓
Thm.2	$O \max\{\sqrt{TV_x}, \sqrt{TV_g}\})$	$O(1)$	✓	✗	✓

- Theorem 1 (Case 2): the regret bound outperforms all existing works

Comparison

Reference	Regret(R)	Constraint violations(C)	Parameter-free	Slater condition-free[1]	Simultaneous sublinear R&C
[Chen et al., 2017]	$O(\max\{V_x T^a, V_g T^a, T^{1-a}\})$	$O(T^{1-a})$	✓	✗	✗
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[Cao and Liu, 2018]	$O(V_x^{\frac{1}{2}} T^{\frac{1}{2}})$	$O(V_x^{\frac{1}{4}} T^{\frac{3}{4}})$	✗	✗	✓
Thm.1	$O(\max\{\sqrt{TV_x}, V_g\})$	$O(\max\{\sqrt{T}, V_g\})$	✓	✓	✓
Thm.1	$O(\sqrt{TV_x})$	$O(\max\{T^{\frac{3}{4}}, V_g\})$	✓	✓	✓
Thm.2	$O \max\{\sqrt{TV_x}, \sqrt{TV_g}\})$	$O(1)$	✓	✗	✓

- Theorem 2: the bound of constraint violations outperforms all existing works

Comparison

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[Chen et al., 2017]	$O(\max\{V_x T^a, V_g T^a, T^{1-a}\})$	$O(T^{1-a})$	✓	✕	✕
[Chen et al., 2018]	$O(T^{\frac{7}{8}} V_x)$	$O(\max\{T^{\frac{15}{16}}, T^{\frac{7}{8}} V_x\})$	✓	✓	✕
[Chen and Giannakis, 2019]	$O(V_x T^{\frac{3}{4}})$	$O(T^{\frac{3}{4}})$	✓	✕	✕
[Chen and Giannakis, 2019]	$O(V_x T^{\frac{1}{2}})$	$O(T^{\frac{1}{2}})$	✓	✕	✕
[Cao and Liu, 2018]	$O(V_x^{\frac{1}{2}} T^{\frac{1}{2}})$	$O(V_x^{\frac{1}{4}} T^{\frac{3}{4}})$	✕	✕	✓
Thm.1	$O(\max\{\sqrt{TV_x}, V_g\})$	$O(\max\{\sqrt{T}, V_g\})$	✓	✓	✓
Thm.1	$O(\sqrt{TV_x})$	$O(\max\{T^{\frac{3}{4}}, V_g\})$	✓	✓	✓
Thm.2	$O\max\{\sqrt{TV_x}, \sqrt{TV_g}\})$	$O(1)$	✓	✕	✓

- Parameter-free, that is, the parameters in our algorithm do not require prior information of the regularities (e.g., V_x or V_g)

Comparison

Reference	Regret(R)	Constraint violations(C)	Parameter-free	Slater condition-free[1]	Simultaneous sublinear R&C
[Chen et al., 2017]	$O(\max\{V_x T^a, V_g T^a, T^{1-a}\})$	$O(T^{1-a})$	✓	✗	✗
[Chen et al., 2018]	$O(T^{\frac{7}{8}} V_x)$	$O(\max\{T^{\frac{15}{16}}, T^{\frac{7}{8}} V_x\})$	✓	✓	✗
[Chen and Giannakis, 2019]	$O(V_x T^{\frac{3}{4}})$	$O(T^{\frac{3}{4}})$	✓	✗	✗
[Chen and Giannakis, 2019]	$O(V_x T^{\frac{1}{2}})$	$O(T^{\frac{1}{2}})$	✓	✗	✗
[Cao and Liu, 2018]	$O(V_x^{\frac{1}{2}} T^{\frac{1}{2}})$	$O(V_x^{\frac{1}{4}} T^{\frac{3}{4}})$	✗	✗	✓
Thm.1	$O(\max\{\sqrt{TV_x}, V_g\})$	$O(\max\{\sqrt{T}, V_g\})$	✓	✓	✓
Thm.1	$O(\sqrt{TV_x})$	$O(\max\{T^{\frac{3}{4}}, V_g\})$	✓	✓	✓
Thm.2	$O \max\{\sqrt{TV_x}, \sqrt{TV_{\boldsymbol{g}}}\})$	$O(1)$	✓	✗	✓

- Holds no matter whether the Slater condition holds or not

Insight

- Connections to queue systems stability.
  - Create a real queue $\boldsymbol{Q}(t)$ to keep track of the “debt” to constraints up to round t
    - $\boldsymbol{Q}(t) = \mathbf{0}, \ \ \mathbf{Q}(\mathrm{t}) = [\mathbf{Q}(\mathrm{t}-1) + \boldsymbol{g}_t(x_t)]^+ = [\sum_{\tau=1}^{t} \boldsymbol{g}_\tau(x_\tau)]^+$

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  - Sublinear constraint violations means the real queue system $\{\boldsymbol{Q}(t)\}$ mean rate stable [Neely, 2010], i.e.,

$$\boxed{\lim_{T\to\infty} \frac{||\boldsymbol{Q}(T)||}{T} = 0}$$

  - $O(1)$ constraint violations is sufficient to show the real queue system $\{\boldsymbol{Q}(t)\}$ strongly stable [Neely, 2010], i.e.,

$$\boxed{\lim_{T\to\infty} \frac{\sum_{t=1}^{T} ||\boldsymbol{Q}(t)||}{T} \leq \mathrm{B} < \infty}$$

Simulation

- Baselines [8, 3, 11, 9]:
  - [Chen et al., 2017, Chen et al., 2018, Cao and Liu, 2019, Chen and Giannakis, 2019]

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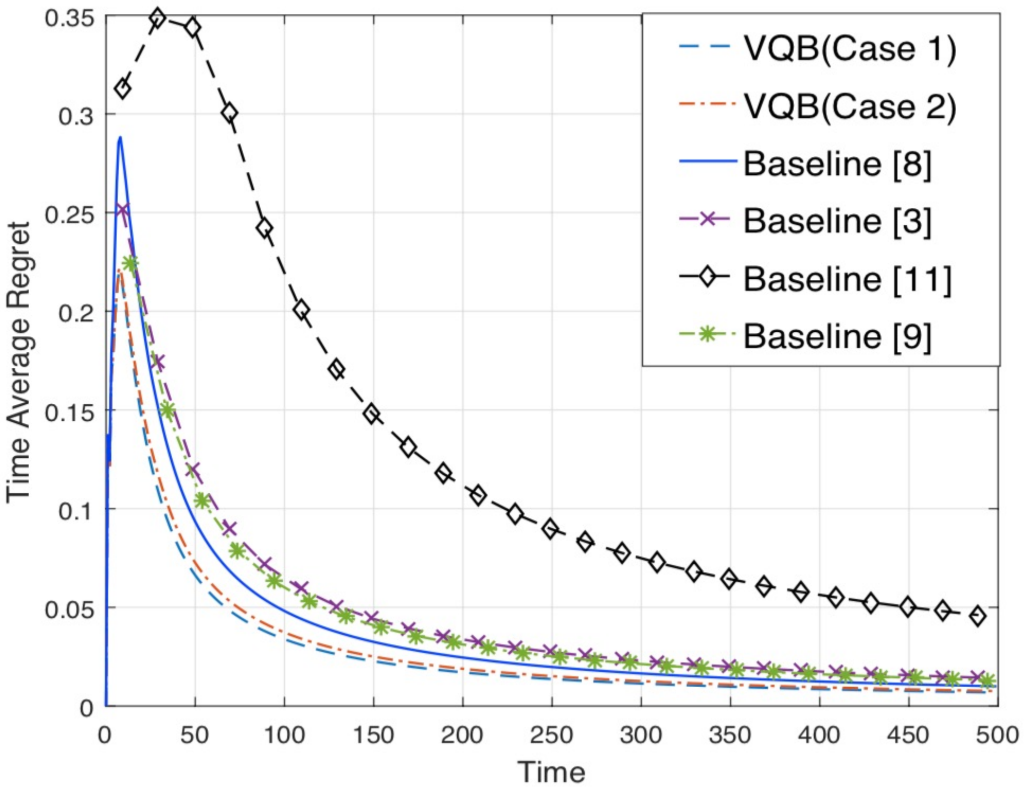
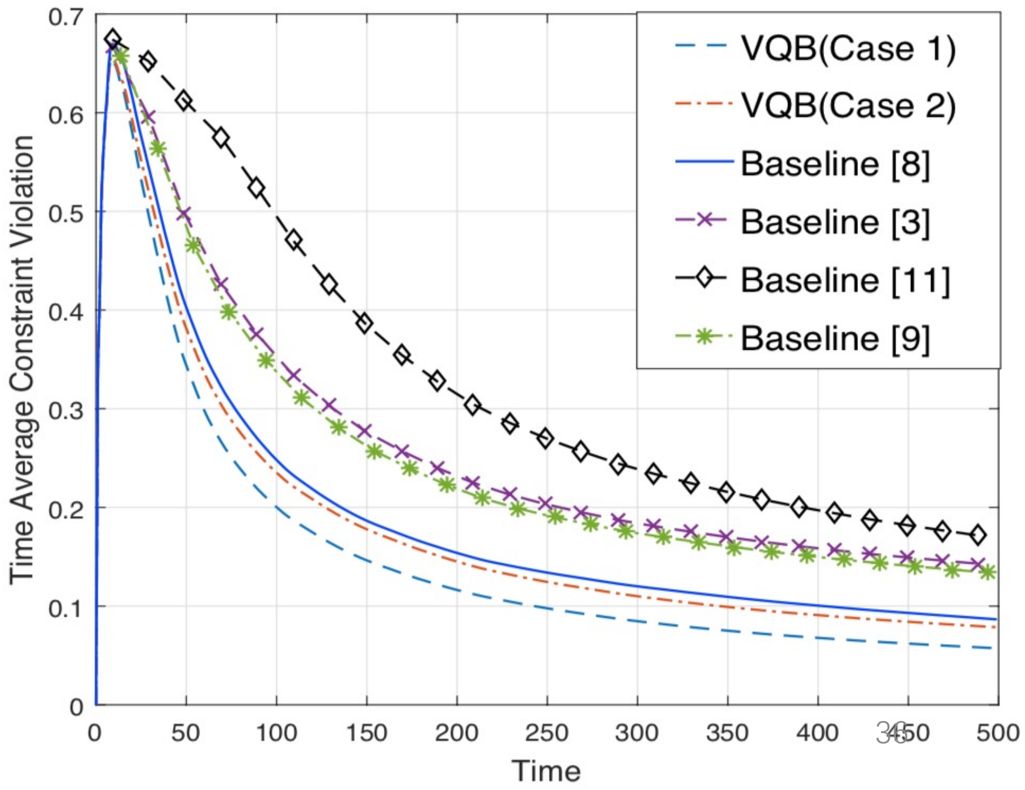
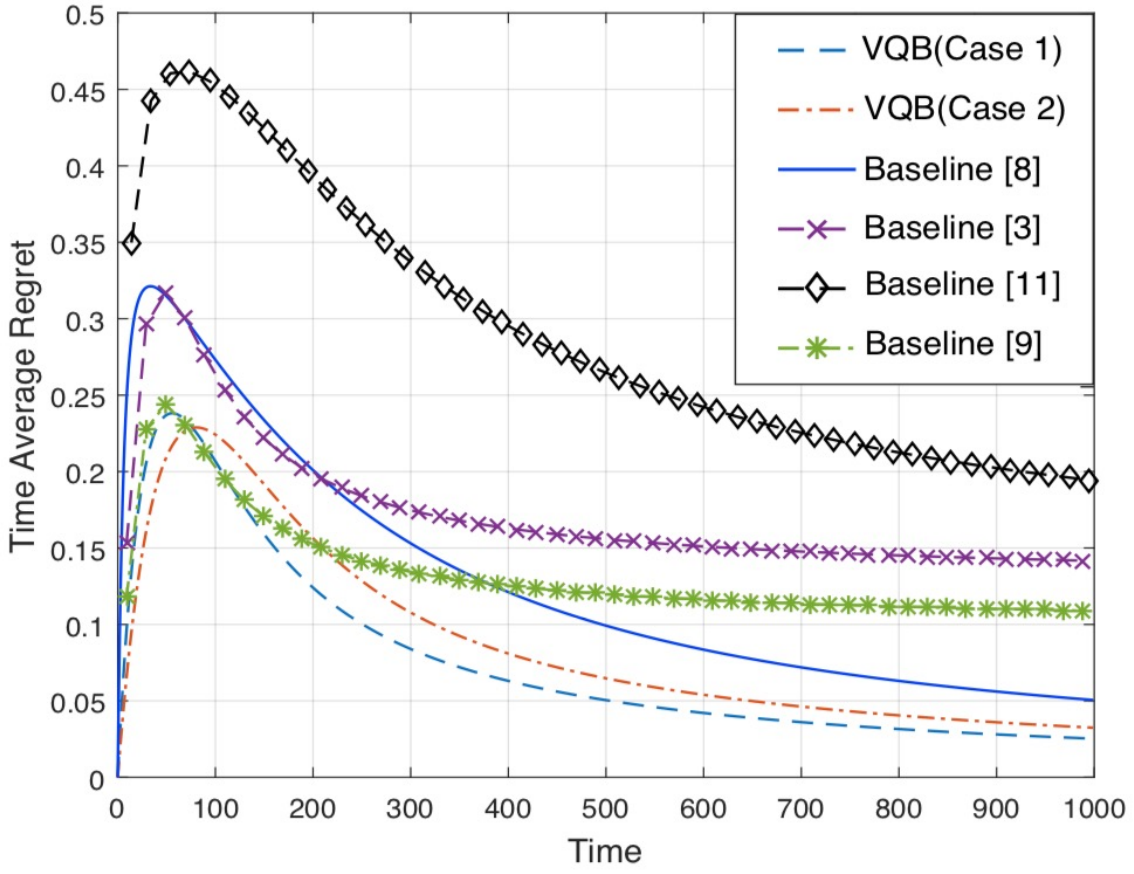
- Setting 1: $V_x = V_g = O(\ln T)$

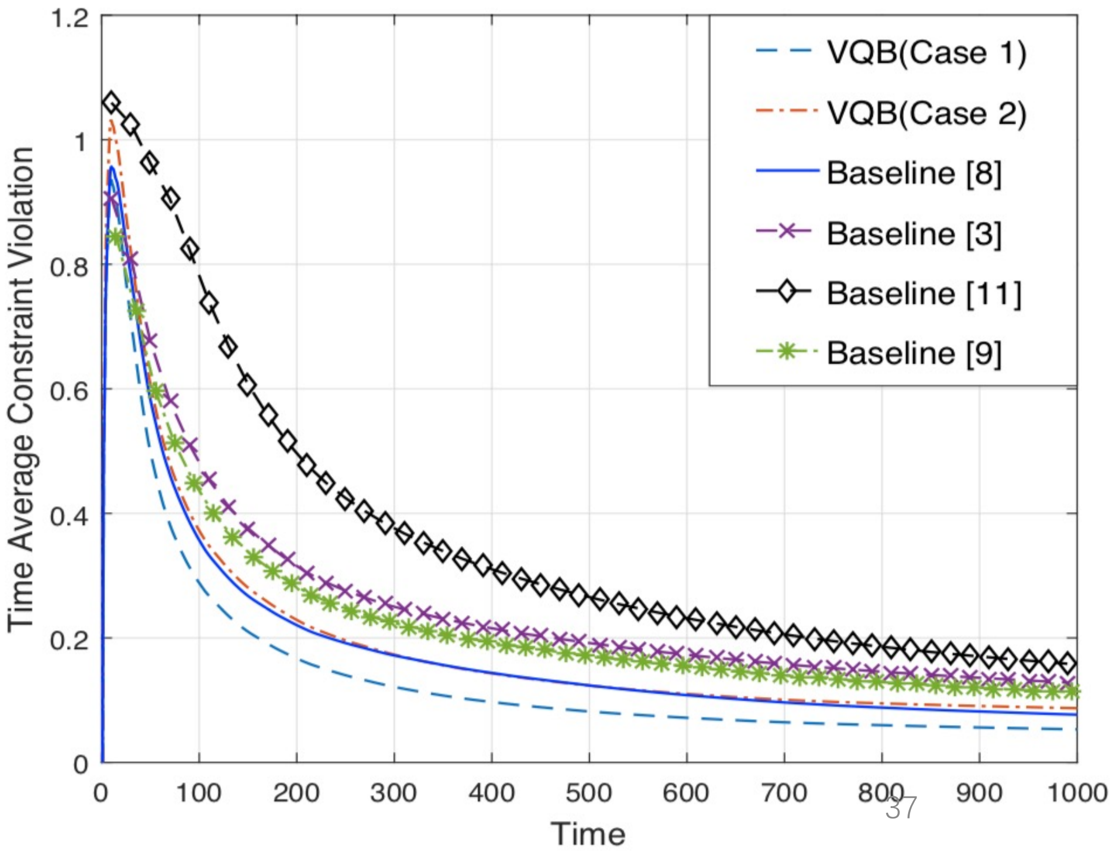




Simulation

- Setting 2: $V_x = V_g = O(\sqrt{T})$





Q & A

Thanks!