Coalition Formation Resource Sharing Games in Networks

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Outline

Motivation

- Limited resources
- Spectral division
- Might be beneficial to collaborate
  - Canada's wireless spectrum auction in 2008
  - Analysts predicted \$2 billion
  - But because of competition - \$4.25 billion
- Selfish players
- Possible presence of an adamant player

Spectrum Share Example

- Player i bid $x_i = \lambda_i a_i$
- Spectrum share $= \frac{\lambda_i a_i}{\sum_{j=0}^{n} \lambda_j a_j}$
- Cost for player i, $\gamma_i = \frac{\gamma}{\lambda_i} x_i$
- The utility of any player i equals

$$\varphi_i = \frac{\lambda_i a_i}{\sum_{j=0}^{n} \lambda_j a_j} - \gamma a_i$$

- People may want to bid together
- Spectrum has to be divided
- Is cooperation beneficial? If yes, then when?
- Which type of players will form coalitions?

Figure: Source: The Economic Times

Problem Description

- A resource sharing game (RSG) with $n+1$ players, $N = \{0, 1, \cdots, n\}$
- The utility of any player i equals

$$\varphi_i = \frac{\lambda_i a_i}{\sum_{j=0}^{n} \lambda_j a_j} - \gamma a_i \; \forall \; i \; \in \; N$$

  where γ = cost factor, λ_i = influence factor and a_i = action of i^{th} player
- Players interested in 'selfish' cooperative opportunities
- Adamant player is not interested in cooperation

Problem Description Contd.

- A Partition, $\mathcal{P}$ is a set of coalitions such that

$$\cup_{i=0}^{k} S_i = N \text{ and } S_i \cap S_j = \emptyset, \text{ null set, } \forall\, i \neq j$$

- Players in *coalition* S_i choose their strategies together
- Utility of a coalition = sum of utilities of its players:

$$\varphi_{S_m}(\mathbf{a}_m, \mathbf{a}_{-m}) = \frac{\sum_{l \in S_m} \lambda_l a_l}{\lambda_0 a_0 + \sum_{l=1}^{n} \lambda_l a_l} - \gamma \sum_{l \in S_m} a_l; \; m \geq 1$$

$$\text{where, } \; \mathbf{a}_m = \{a_i, i \in S_m\}, \mathbf{a}_{-m} = \{a_i, i \notin S_m\}$$

- **AIM:** to study 'partition of coalitions' that is 'stable'

Coalition Formation Game Ingredients

Set of players

Players interested in forming coalitions, $N_C = \{1, \cdots, n\}$.

Strategy[1]

- A strategy of a player = coalition formation interests, $x_i \subseteq N_C$
- The strategy set of a player i, denoted by X_i is defined as:

$$X_i = \{x_i : i \in x_i \text{ and } x_i \subseteq N_C\}$$

- $i \in x_i$ in all strategies

Utility of each player for any strategy profile $\underline{x} = \{x_i\}_i$?

[1] S. Nevrekar. A theory of coalition formation in constant sum games, 2015.

Ingredients: Possible partitions, given interests/strategy profile

Partition formed under strategy profile $\underline{x}$

- Coalition $S \in \mathcal{P}(\underline{x})$, if it satisfies:

$$i \in x_j \text{ and } j \in x_i \text{ for all } i, j \in S;$$

  i and j are in same coalition only if there is a mutual interest!

- There exists no other partition $\mathcal{P}'$

$$S_i \text{ and } S_j \in \mathcal{P} \text{ such that } S_i \cup S_j \subseteq S' \in \mathcal{P}'$$

  preference is given to coarser partitions!

A strategy profile $\longrightarrow$ multiple partitions

For example, $x_1 = \{1, 2\}, x_2 = \{1, 2, 3\}$ and $x_3 = \{1, 2, 3\}$

$$\mathcal{P}_1 = \{\{1, 2\}, \{3\}\} \text{ and } \mathcal{P}_2 = \{\{1\}, \{2, 3\}\}$$

Ingredients: Resource sharing game among coalitions in $\mathcal{P}$

Resource Sharing Game (RSG)

- Coalitions are players, utility = sum utility
- $\bar{\lambda}_i^{\mathcal{P}}$ - highest influence factor among players in coalition S_i

Theorem: Utilities of coalitions at RSG-Nash Equilibrium [2]

- Possibility of many NE, but unique NE-utilities
- Only top coalitions (larger $\bar{\lambda}_i^{\mathcal{P}}$) derive non-zero utility ($\mathcal{J}^*$)
- Unique NE-utility of coalition

$$\varphi^*_{S_m}(\mathcal{P}) = \left(\frac{s^{\mathcal{P}} - \frac{M^{\mathcal{P}}-1}{\bar{\lambda}_m^{\mathcal{P}}}}{s^{\mathcal{P}}} \right)^2 \mathbb{1}_{S_m \in \mathcal{J}^*}, M^{\mathcal{P}} := \max \left\{ m \leq k : \sum_{i=1}^{m} \frac{1}{\bar{\lambda}_i^{\mathcal{P}}} - \frac{m-1}{\bar{\lambda}_m^{\mathcal{P}}} > 0 \right\}, s^{\mathcal{P}} = \sum_{m=1}^{M^{\mathcal{P}}} \frac{1}{\bar{\lambda}_m^{\mathcal{P}}}$$

[1] R. Dhounchak, V. Kavitha, and Y. Hayel. To participate or not in a coalition in adversarial games. In NetGCooP.

Ingredients: Division of coalitional worth among its members

Division of worth in any $\mathcal{P} = \{S_1, S_2, \cdots, S_k\}$: Extension of Shapley value[3]

- For division within a coalition (S_i)- S_i is considered as grand coalition.
- Usual Shapley value definition is used

$$\phi_j^*(\mathcal{P}) = \sum_{C \subseteq S_i, j \notin C} \frac{|C|!(|S_i| - |C| - 1)!}{|S_i|!} \left[\nu^{\mathcal{P}}_{C \cup \{j\}} - \nu^{\mathcal{P}}_C\right] \text{ and } j \in S_i$$

- $\nu_C^{\mathcal{P}}$ = worth of sub-coalition C = pessimal utility at RSG-NE
  - Environment $(S_1, \cdots, S_{i-1}, S_{i+1}, S_k)$ is assumed to be fixed.
  - Other members, $S_i - C$ choose to hurt C the most [3, 2]

Symmetric players $\implies$ equal shares

[1] R. J. Aumann and J. H. Dreze. Cooperative games with coalition structures. International Journal of game theory, 3(4):217–237, 1974.

Ingredients: Player utilities for given strategy profile

Lemma

- *Pessimal utility of sub-coalition, C $(\nu_C^{\mathcal{P}})$ = obtained when players in $S_i - C$ are arranged as singletons in RSG.*

Utility of a player = worst utility under all possible partitions $\{\mathcal{P}(\underline{x})\}$

$$U_i(\underline{x}) = \min_{\mathcal{P}(\underline{x})} \phi_i^*(\mathcal{P}(\underline{x}))$$

Thus we have a strategic form game !

AIM: to study the Nash Equilibrium (NE) and the partitions emerging at NE

Solution Concepts

NE-partitions = Partitions at Nash Equilibrium

- Nash Equilibrium $\underline{x}^*$: strategy profile where no player can deviate unilaterally and obtain higher utility.
- NE-partitions = $\{\mathcal{P}(\underline{x}^*)\}$

$x_j^{\mathcal{P}} = S_i$ for any $j \in S_i \in \mathcal{P}$ = natural strategy profile that uniquely leads to $\mathcal{P}$.

U-stable partitions

A partition $\mathcal{P}$ is said to be a *U-stable partition if the corresponding natural strategy profile* $\underline{x}^{\mathcal{P}}$ *is a Nash Equilibrium.*

- All U-stable partitions are NE-partitions but the vice-versa may not be true!

Social Optima (SO)-partitions

Maximises the sum utility of all players in N_C.

Results: Symmetric Players for $\lambda_i = \lambda$

$\eta = \lambda_0/\lambda =$ relative strength of adamant player

SO-Partitions

η

0 ← Grand Coalition → 0.414 ← Duopoly → 0.707 ← Grand Coalition →

- Under Grand coalition, adamant player always derives positive utility
- Else, it derives zero utility when $\eta \leq (n-1)/n$.

Theorem (Unique NE for symmetric players: when $n > 4$)

All players alone (ALC) is the unique NE-partition.

For $n \leq 4$, results in paper.

Results with symmetric players

Price of Anarchy: Estimate loss of players due to their 'selfishness'

Ratio of sum utilities at 'social optima' and sum utilities at 'worst Nash Equilibrium'.

NE-Partition= ALC, SO-Partitions and P_{oA} for $n > 4$

	$[0, 0.414]$	$[0.414, 0.5]$	$[0.5, 0.707]$	$[0.707, \frac{n-1}{n}]$	$[\frac{n-1}{n}, \infty)$
P_{oA}	$\frac{n}{(1+\eta)^2}$	$\frac{n}{2}$	$\frac{2n}{(1+2\eta)^2}$	$\frac{n}{(1+\eta)^2}$	$\frac{1}{n}\left[\frac{1+n\eta}{1+\eta}\right]^2$
SO-Partitions	GC	GC	Duopoly	Duopoly	GC

η: 0, 0.414, 0.5, 0.707, $\frac{n-1}{n}$

- P_{oA} increases as $O(n)$ when $n \to \infty$.
- As $\eta \to 0$ or ∞, $P_{oA} \uparrow n$.

Single Asymmetric Player

- One asymmetric player with influence factor $\beta\lambda$ (with $\beta > 1$)
- n symmetric players with influence factor λ

Theorem (Under certain assumptions)

For $n > 5$ and $\beta > 1$, the only U-stable partitions are of the form

- *Coalition with asymmetric player can contain more players*
- *Others are all alone*

One asymmetric player $\Longrightarrow$ More stable partitions!

Absolute Stability

Assuming $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Definition: Absolute Stability

All partitions are U-stable.

Definition: Measure of asymmetry

$$\mathcal{A}_I := \min_{2 \leq j \leq n} \frac{\varrho_{j+1}^2 - \varrho_j^2}{c_j (1 - \varrho_j)^2}, \text{ where } \varrho_j = \lambda_1 / (\lambda_1 + \lambda_j) \text{ and } c_j := 1_{j=2} + j 1_{j>2}$$

Theorem (When player 1 is significantly influential than others)

The system is absolutely stable if and only if $\mathcal{A}_I \geq 1$.

Moderate Asymmetry

- $\lambda_1 \geq \cdots \geq \lambda_{j-1} \geq \lambda_j \geq \cdots \geq \lambda_k \geq \lambda_{k+1} \geq \cdots \geq \lambda_n$
- SS($\{j,k\}$) partition $\implies$ players other than j and k are alone.

Theorem (Bigger players, higher chance to form coalitions)

Under certain conditions, partition SS($\{j-1,k\}$) is stable for any $j<k$, if partition SS($\{j,k\}$) is stable.

Theorem (Smaller players, higher chance to form coalitions)

The partition SS($\{j,k+1\}$) is stable for any $j<k$, if partition SS($\{j,k\}$) is stable.

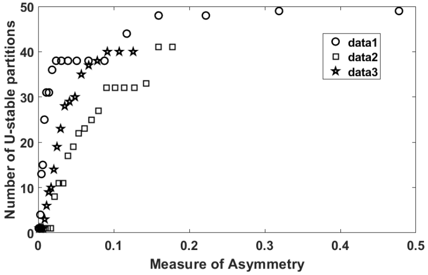

Figure: $\lambda_j = 20 - \alpha_j \delta$, $\alpha_j \sim U(0,1)$ and $\delta \in [1, 20]$

- Number of U-stable partitions increases as measure of asymmetry increases.

Numerical Results Contd.

δ	No. of Stable partitions			Additional Stable Partitions
	ALC	TTC	SS	
0.1	1	0	0	$\{\{1\},\{2\},\{3\},\{4\},\{5\}\}$
0.146	1	0	1	$\{\{5,1\},\{4\},\{3\},\{2\}\}$
0.147	1	2	1	$\{\{5,3\},\{4,1\},\{2\}\}$ $\{\{5,2\},\{4,1\},\{3\}\}$
0.18	1	2	2	$\{\{5,2\},\{4\},\{3\},\{1\}\}$
0.19	1	2	3	$\{\{5,3\},\{4\},\{2\},\{1\}\}$

Table: $\lambda_j = 20 - \alpha_j\delta$, $\alpha = [0,\ 8,\ 11.5,\ 15.3,\ 21.5]$

- With almost equal players - only ALC
- Highest and lowest players are the first ones to collaborate
- If SS($\{j,k\}$) is stable, then SS($\{j-1,k\}$), SS($\{j,k+1\}$) and SS($\{j-1,k+1\}$) are also stable

Spectral Share Example

Stable Configurations	Partition
1	$\{30, 30, 35, 35\}$
2	$\{\{30, 35\}, \{30, 35\}\}$
3	$\{\{30, 30\}, \{35, 35\}\}$
4	$\{\{30, 35\}, \{30\}, \{35\}\}$
5	$\{\{30\}, \{30\}, \{35\}, \{35\}\}$

Table: Partitions described by stable configurations

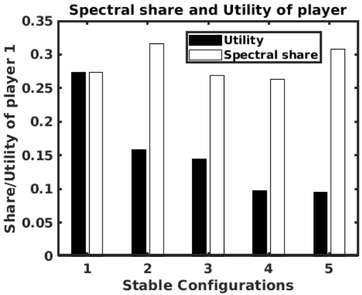

Figure: $\lambda_j = [35, 35, 30, 30]$ and $\gamma = 1$

- Highest spectral share - at config. 2
- Highest utility - at config. 1
- Asymmetric players together in config. 2 - performs better than config. 3

Conclusions

- With equal or almost equal players, no one collaborates at equilibrium if $n > 4$.
- High price of anarchy.
- Identified conditions for absolute stability, for the case of asymmetric players.
  - Absolute stability = all partitions stable
- Stable partitions (against unilateral deviations) increase as asymmetry increases.
- The highest and the lowest capacity players are the first ones to collaborate.

Coalitional Stability

- GC - under absolute stability conditions
- No partition - symmetric players

References

[1] R. J. Aumann and J. H. Dreze. Cooperative games with coalition structures. International Journal of game theory, 3(4):217–237, 1974.

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[3] Bloch, Francis and Van den Nouweland, Anne. Expectation formation rules and the core of partition function games, Games and Economic Behavior, 2014.

[4] R. Dhounchak, V. Kavitha, and Y. Hayel. To participate or not in a coalition in adversarial games. In NetGCooP.

[5] S. Nevrekar. A theory of coalition formation in constant sum games, 2015.

Thank you