A Heavy Traffic Theory of Two-Sided Queues

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Introduction

Marketplaces

Traditional Marketplace

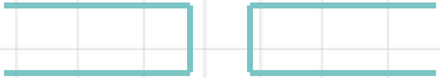
Fundamental Model:
Single Sided Queue

Applications

Literature dates back 100 years [Erlang, 1909]

Gig Marketplace

Fundamental Model:
Two-Sided Queue



Applications

Many Emerging Applications [2000]

Literature Review

Many related models in the literature:

- **Bipartite Matching Models** [Adan, Weiss, 2012], [Caldeney et. Al. 2009], [Adan et. al. 2018], [Cadas et. al. 2019]
- **Matching Models** [Mairesse, Moyal, 2016], [Cadas et. al. 2020], [Moyal, Perry, 2017]
- **Matching Queues** [Gurvich, Ward, 2014]
- **Assemble to Order Systems** [Song, Zipkin, 2003], [Song, 1998], [Song et. al. 1999], [Song, 2002], [Song, Yao, 2002], [Plambeck, Ward, 2006], [Dogru et. al. 2010]
- **Other Related Models** [Anderson et. al.], [Akbarpour et. al. 2019]
- **Two-Sided Queues with few differences** [Hu, Zhou, 2018], [Nguyen, Stolyar, 2018], [Aveklouris et. al. 2021], [Ozkan, Ward, 2017], [Ozkan, 2020], [Blanchet, et. al. 2021]

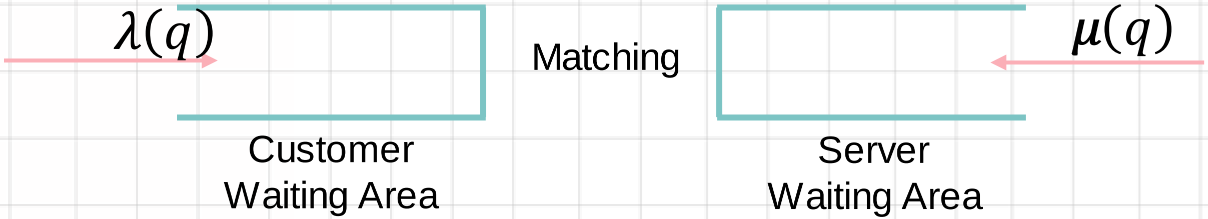
Most models where the system is inherently unstable, only transience analysis have been done except

[Nguyen, Stolyar, 2018], [Blanchet, et. al. 2021] [Varma, et. al. 2020, 2021]

We conduct more fine-tuned analysis

Model and Results

Model



Stability

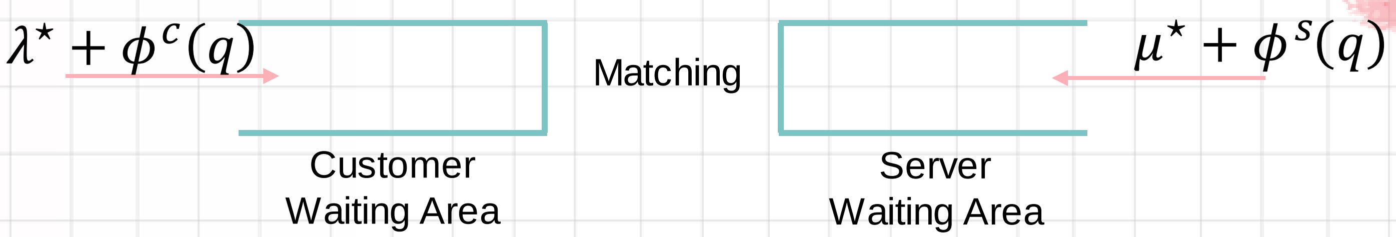
$\lambda > \mu$	Transient
$\lambda = \mu$	Null Recurrent
$\lambda < \mu$	Transient

Need External Control to make the System Stable

Dynamic Pricing

Can be Analyzed in Steady State

Model



Discrete Time

State: Imbalance (z)

Arrivals:

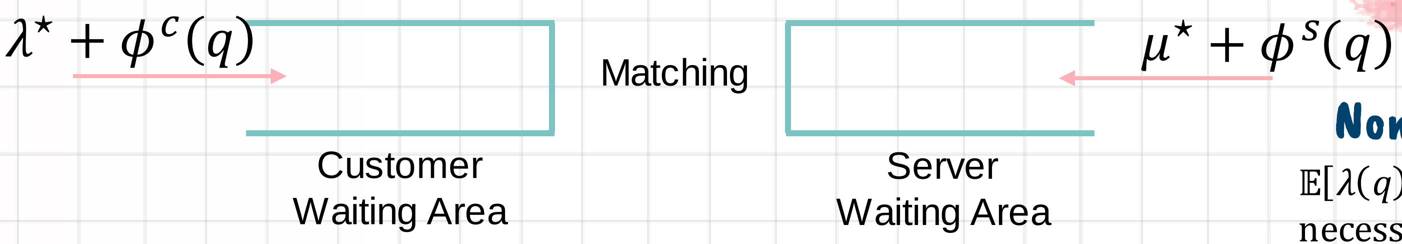
$$a^{c} \sim General\left(\lambda(z), \sigma^{c}(\lambda(z))\right)$$

$$a^{s} \sim General\left(\mu(z), \sigma^{s}(\mu(z))\right)$$

Queue Evolution:

$$z(k+1) = z(k) + a^{c}(k) - a^{s}(k)$$

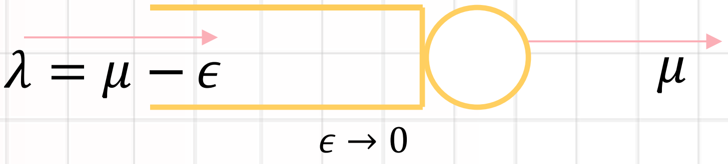
Heavy Traffic in Two-Sided Queues



Non-Trivial!

$\mathbb{E}[\lambda(q)] = \mathbb{E}[\mu(q)]$ is necessary for stability

Heavy Traffic in Single Sided Queue



Two Interpretations:

1. External Control Vanishes
2. Phase Transition from Positive to Null Recurrent

Heavy Traffic is defined as the limit:

$$\phi^c(\cdot) \xrightarrow{u} 0, \phi^s(\cdot) \xrightarrow{u} 0$$

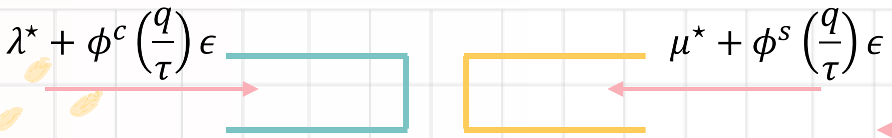
$$\lambda(q) = \lambda^\star + \phi^c\left(\frac{q}{\tau}\right)\epsilon$$

$$\mu(q) = \mu^\star + \phi^s\left(\frac{q}{\tau}\right)\epsilon$$

$$\lambda^\star = \mu^\star$$

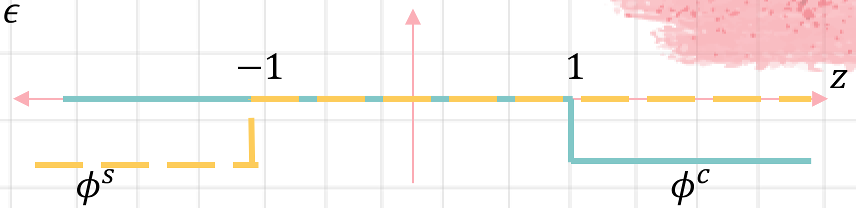
$$\epsilon \to 0, \tau \to \infty$$

Illustrative Example



$$a^s(q) \sim Bernoulli\left(\mu^\star + \phi^s\left(\frac{q}{\tau}\right)\epsilon\right)$$

$$a^c(q) \sim Bernoulli\left(\lambda^\star + \phi^c\left(\frac{q}{\tau}\right)\epsilon\right)$$



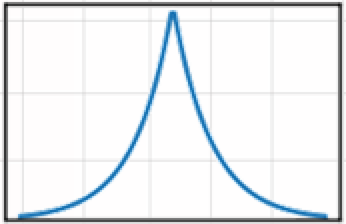
A Birth-Death Process

Case I: $\epsilon\tau \to 0$

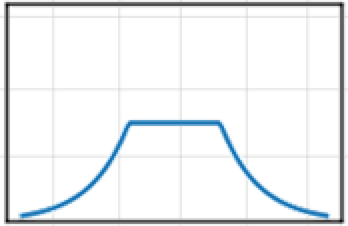
Case II: $\epsilon\tau \to (0, \infty)$

Case III: $\epsilon\tau \to \infty$

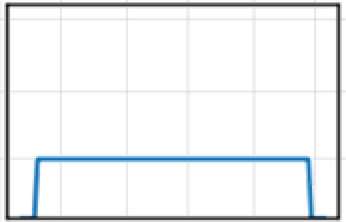
$\epsilon z \to Laplace$



$\epsilon z, \frac{z}{\tau} \to Hybrid$

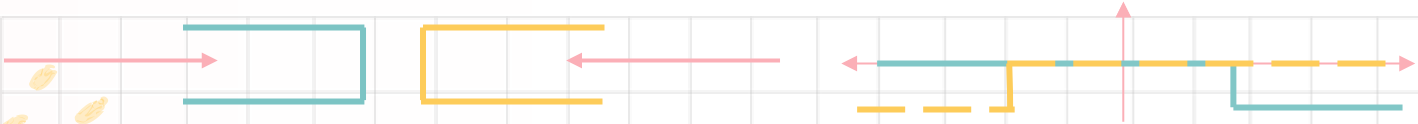


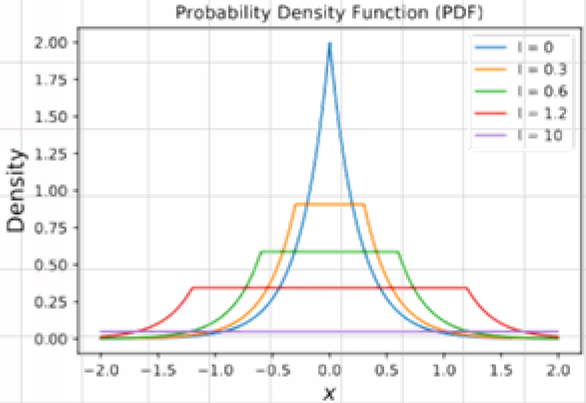
$\frac{z}{\tau} \to Uniform$



A Phase Transition from Laplace to Uniform

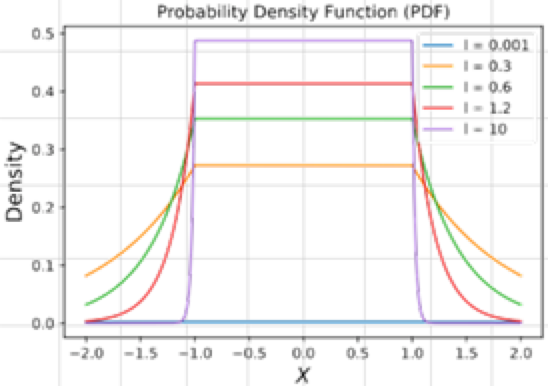
Further Intuition





ϵz

$\epsilon\tau \to l$



$\frac{z}{\tau}$

Hybrid -> Laplace as $l \to 0$

Hybrid -> Uniform as $l \to \infty$

General Case

Arrival Rates:

$$\lambda(q) = \lambda^\star + \phi^c\left(\frac{q}{\tau}\right)\epsilon$$
$$\mu(q) = \mu^\star + \phi^s\left(\frac{q}{\tau}\right)\epsilon$$

Arrivals:

$$a^c \sim General\left(\lambda(q), \sigma^c(\lambda(q))\right)$$
$$a^s \sim General\left(\mu(q), \sigma^s(\mu(q))\right)$$

Assumptions:

$\phi^c(\cdot)$ and $\phi^s(\cdot)$ are bounded

$$\phi^c(x) - \phi^s(x) \begin{cases} < -\delta \; for \; x > K \\ > \delta \; for \; x < -K \end{cases}$$

Theorem 1 ($\epsilon\tau \to l \in (0,\infty)$):

$$\epsilon z \to Gibbs(g_1) \text{ where } g_1(x) = \frac{2}{\sigma^c(\lambda^\star) + \sigma^s(\mu^\star)}\left(\phi^s\left(\frac{x}{l}\right) - \phi^c\left(\frac{x}{l}\right)\right)$$

$$\frac{z}{\tau} \to Gibbs(g_2) \text{ where } g_2(x) = \frac{2l}{\sigma^c(\lambda^\star) + \sigma^s(\mu^\star)}\left(\phi^s(x) - \phi^c(x)\right)$$

$$Gibbs(g) = Ce^{-\int_0^x g(t)\,dt}$$

As $l \to 0$, $g_1(x) \to \frac{2}{\sigma^c(\lambda^\star)+\sigma^s(\mu^\star)}\left(\phi^s(\infty\,\mathrm{sgn}(z)) - \phi^c(\infty\,\mathrm{sgn}(z))\right)$

Thus, $Gibbs(g_1) \to Laplace$ as $l \to 0$

l is the so called 'temperature' of $Gibbs(g_2)$

As $l \to \infty$, the PDF of $Gibbs(g_2)$ vanishes everywhere except $\int_0^x \phi^s(t) - \phi^c(t)\; dt$ attains its minimum

Theorem 2 ($\epsilon\tau \to 0$):

$$\epsilon z \to Laplace$$

Conjecture 3 ($\epsilon\tau \to \infty$):

$$\epsilon z \to Uniform(\Phi^\star) \text{ where } \Phi^\star = \arg\min \int_0^x \phi^s(t) - \phi^c(t)\; dt$$

Proof Idea

Inverse Fourier Transform Method

Tightness

Ensures existence of z_∞ such that

$$\epsilon z \to z_\infty$$

Inverse Fourier Transform

Let $f(\cdot)$ be the PDF of z_∞

$$f'(z_\infty) = f(z_\infty)g(z_\infty)$$

IFT

Derivative Theorem for IFT

Formalized using Schwartz functions and Tempered Distributions

Set Drift of $e^{j\omega\epsilon z}$ to Zero

$$\mathbb{E}\left[e^{j\omega\epsilon z(k+1)}\right] - \mathbb{E}\left[e^{j\omega\epsilon z(k)}\right] = 0$$

$\epsilon \to 0, \tau \to \infty \; s.t. \; \epsilon\tau \to (0,\infty)$

$$j\omega\mathbb{E}\left[e^{j\omega z_\infty}\right] = \mathbb{E}\left[e^{j\omega z_\infty} g(z_\infty)\right]$$

$$\Updownarrow$$

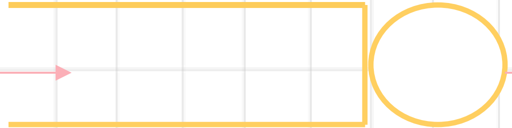
$$j\omega\mathcal{F}[f(z_\infty)] = \mathcal{F}[f(z_\infty)g(z_\infty)]$$

Implicit equation involving the characteristic function

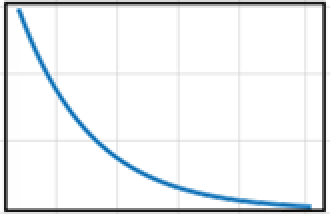
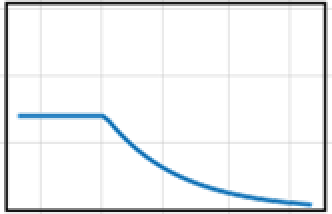
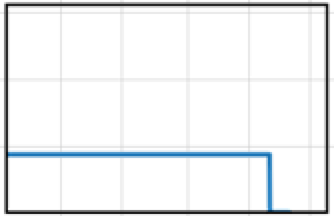
Single Server Queue

Single Server Queue

$\lambda^\star + \phi^c\left(\frac{q}{\tau}\right)\epsilon$



$\mu^\star + \phi^s\left(\frac{q}{\tau}\right)\epsilon$

	$\epsilon\tau \to 0$	$\epsilon\tau \to (0,\infty)$	$\epsilon\tau \to \infty$
Bernoulli Arrivals Two-Price Policy	 Exponential	 Hybrid	 Uniform
General Arrivals General Arrival Rates	Exponential	One Sided Gibbs	Uniform (Conjecture)

Thanks!

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