The CME method: Efficient numerical inverse Laplace transformation with Concentrated Matrix Exponential distribution

Salah Al-Deen Almousa[⋆], Gábor Horváth[⋆], Illés Horváth[†], András Mészáros[⋆], *Miklós Telek*[⋆†]

[⋆]Budapest University of Technology and Economics, Hungary
[†]MTA-BME Information Systems Research Group, Hungary

TOSME – Tools for Stochastic Modelling and Evaluation
Performance 2021 workshop

Nov. 12, 2021

The CME method: Simple, reliable numerical inverse Laplace transformation

Salah Al-Deen Almousa⋆, Gábor Horváth⋆, Illés Horváth†, András Mészáros⋆, *Miklós Telek*⋆†

⋆Budapest University of Technology and Economics, Hungary
†MTA-BME Information Systems Research Group, Hungary

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Outline

Numerical inverse Laplace transformation (NILT):

- numerically sensitive,
- mysterious dependence on the order,
- Gibbs oscillation.

Laplace transformation

Laplace transform is defined as

$$h^*(s) = \int_{t=0}^{\infty} e^{-st} h(t) \mathrm{d}t.$$

Use of Laplace transform

- Stochastic models,
- Differential equations,
- Electric circuit theory,
- ...

Inverse Laplace transformation

Find $h(t)$ based on $h^*(s)$

- symbolic methods,
- numeric methods (NILT).

NILT: Approximate $h(t)$ at point T (i.e., $h(T)$) based on $h^*(s)$.

Currently dominant approach is the *Abate-Whitt framework*:

- Euler,
- Talbot,
- Gaver-Stehfest,
- CME

W. Whitt J. Abate., A unified framework for numerically inverting Laplace transforms. *INFORMS Journal on Computing*, 18(4):408–421, 2006.

Abate-Whitt framework

Approximate $h(T)$ by a finite linear combination of the transform values, via

$$h(T) \approx h_n(T) := \sum_{k=1}^{n} \frac{\eta_k}{T} h^*\left(\frac{\beta_k}{T}\right),$$

where the *nodes* β_k and *weights* η_k are (potentially) complex numbers, which

- depend on the order n, and the method, but
- do not depend on $h^*(s)$ or the parameter of interest T.

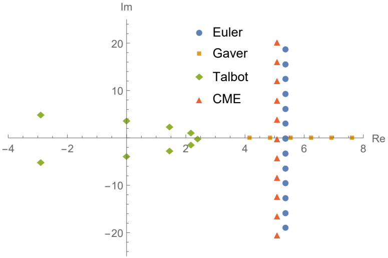
Approaches to obtain nodes β_k and weights η_k (1)

Bromwich inversion formula by the contour integral

$$h(T) = \oint_s e^{sT} h^*(s) ds \approx \sum_{k=1}^{n} \frac{\eta_k}{T} h^*\left(\frac{\beta_k}{T}\right).$$

Used in the

- Euler,
- Talbot,
- Gaver-Stehfest methods.



Approaches to obtain nodes β_k and weights η_k (2)

Integral interpretation:

$$h_n(T) = \sum_{k=1}^{n} \frac{\eta_k}{T} h^*\left(\frac{\beta_k}{T}\right) = \sum_{k=1}^{n} \frac{\eta_k}{T} \int_0^\infty e^{-\frac{\beta_k}{T}t} h(t)\mathrm{d}t$$

$$= \sum_{k=1}^{n} \eta_k \int_0^\infty e^{-\beta_k t} h(tT)\mathrm{d}t = \int_0^\infty h(tT) f_n(t)\mathrm{d}t,$$

i.e., $h_n(T)$ is the integral of $h(tT)$ with the *weight function*

$$f_n(t) = \sum_{k=1}^{n} \eta_k e^{-\beta_k t}.$$

If $f_n(t)$ was the Dirac impulse function at one then the Laplace inversion would be perfect, that is $h_n(T) = h(T)$.

CME method

The CME method is based on the integral interpretation.

Nodes β_k and weights η_k are set to approximate the unit impulse as closely as possible.

$$\min_{\eta_k,\beta_k,k\in\{1,\ldots,n\}} \mathrm{SCV}\left(f_n(t)\right)$$
$$\text{subject to } f_n(t) \geq 0$$

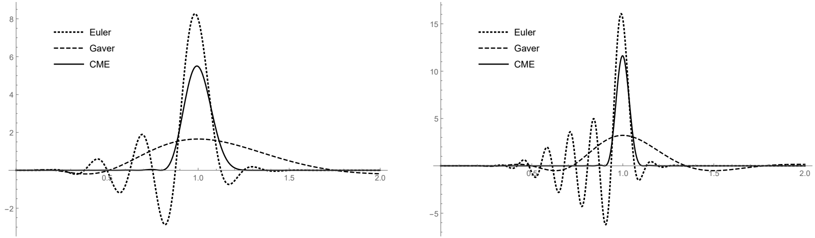
where

$$\mathrm{SCV}\left(f_n(t)\right) = \frac{\int_{t=0}^{\infty} t^2 f_n(t)\mathrm{d}t \int_{t=0}^{\infty} f_n(t)\mathrm{d}t}{\left(\int_{t=0}^{\infty} t f_n(t)\mathrm{d}t\right)^2} - 1$$

G. Horváth, I. Horváth, M. Telek, High order concentrated matrix-exponential distributions. *Stochastic Models*, 36(2):176–192, 2020.

CME versus Euler and Gaver

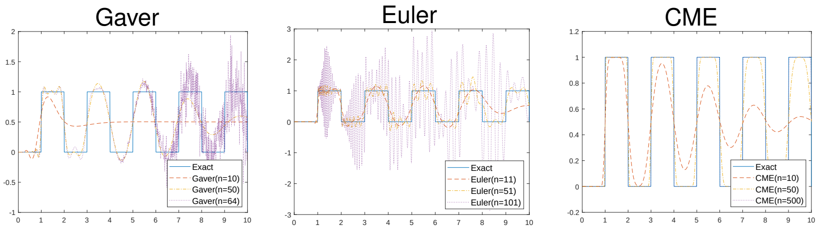
$f_n(t)$ for order 10 and 20



For the CME method $f_n(t) \geq 0$!!

Numerical experiment

Inverse Laplace transformation of the $h(t) = \lfloor t \rfloor \mod 2$ function



Numerical instability | Amplified Gibbs oscillation | Gradually improving

Tool support

Tool support

- Online: `https://inverselaplace.org`
- Offline: `https://github.com/ghorvath78/iltcme`

  Ready to use packages:
  - Python (numpy)
  - Matlab
  - Mathematica

Precomputed nodes β_k and weights η_k are provided in `json` file for order $1 - 1000$.

Summary

The CME method is a member of the Abate–Whitt framework:

- simple and cheap computation,
- but it differs from other AWF methods
  - + improves with increasing order,
  - + $f_n(t)$ is non-negative $\rightarrow$ no Gibbs oscillation,
  - + numerically stable up to $n = 1000$ with double precision arithmetic,
  - ◦ nodes β_k and weights η_k are computed a priori.

Safest choice for “plug and play” application!

Implementation and technical details:

`https://inverselaplace.org`

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